Social Planners and Local Conventions

Fei Shi*

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Abstract

We develop a two-stage, two-location model to investigate the interrelationship between policy-making and social conventions. Rational social planners determine the maximum capacity and mobility constraints. Then, boundedly rational residents play a coordination game with the possibility to migrate. We show that if both planners are only concerned with efficiency, the symmetric policy settings in Ely (2002) and Anwar (2002) are not stable. If they also care about scale, they will completely restrict mobility, and then the risk-dominant equilibrium prevails globally.

Keywords: location models; conventions; social planners; stochastic learning

JEL classification: C72; C73; D83

1 Introduction

In an organized society, social conventions, as a result of the long-run interactions among decision-makers, both influence the development of policies, regulations, and laws, and are in turn, influenced by these same factors. When designing a policy, policy-makers have to understand the behavior of policy-takers and the long-run consequences of the interactions among them, for, to realize the same objective, different behavior may lead to different policy choices. Reciprocally, an effective policy always determines the set of strategies available to the policy-takers, through which it forges social conventions consequently.

For instance, walking along one side of the road is a consequence of long-run interactions among people. However, which side to walk along is regulated by laws that usually, but not universally, follow the local conventions. As another example, a single computer operating system is usually used within a firm. However, which one to use is affected by firm policies. When the firm manager decides not to buy the latest Windows system in order to save on expenditures, the employees may coordinate by themselves to use some free operating system.

The interrelationship between policies and conventions will become more complex if it involves multiple societies with interdependent interests. In this situation, policy-makers have to consider both the effect of the local policy and the policies in other societies. For example, governments of all major countries in the world have migration policies. A specific policy will not only affect the interaction pattern of the common public in this country alone but also that in the neighboring countries. If the interaction patterns are relevant to the interests of the governments, the migration policies of these related countries will be influenced as well. Similarly, in a market with several competing firms, the employment policy of one firm may affect the way that employees coordinate within each firm, and subsequently the profits of the respective firms. Hence, firm managers will choose an optimal policy to maximize profit, while taking the policies of other firms into account.

*University of Konstanz, Department of Economics, Box 150, 78457 Konstanz, Germany. Tel.: +49 (0) 7531 884048; fax: +49 (0) 7531 884119. E-mail: fei.shi@uni-konstanz.de. The author thanks Carlos Alós-Ferrer, Johannes Kern, Larry Blume, and the participants in the 11th SAET conference, the 6th Spain-Italy-Netherlands Meeting on Game Theory, and the seminar at the University of Konstanz for helpful suggestions and comments. Financial support from the Center for Psychoeconomics at the University of Konstanz is gratefully acknowledged.

In general, the interaction among policy-makers has strategic elements. They attempt to choose an optimal policy given their knowledge of the long run consequences of the interactions among policy-takers. After that, policy-takers interact with each other repeatedly by choosing strategies allowed by the policies.

In this context, it is reasonable to consider policy-makers as being more sophisticated ("more rational") than policy-takers. When it comes to countries, governments can be treated as rational, since they can collect detailed information and rely on expert advice when making decisions. Turning to companies, a similar reasoning leads us to consider managers as rational agents. In contrast, policy-takers are always regarded as boundedly rational, either because they do not have an access to enough information, or because they are not able to process information correctly (or simply do not devote enough resources to process it). Usually, they adopt certain rule of thumb to guide their behavior. Hence, agents with heterogeneous rationality levels interact with each other within this framework.

The interaction between policies and social conventions has been pointed out e.g. by Young (1998). There are, however, few studies that *explicitly* model this interaction. How will policy-makers use policy instruments to affect social conventions? How will the characteristics of the interactions among policy-takers influence policy-makers' choices? What will be the end result given this mutual influence? Will a convention prevail only locally, or spread globally? This paper seeks to investigate these questions and provide new insights from an evolutionary game-theoretical point of view.

To focus on the pure interaction between policy-making and social conventions within a theoretical framework, we single out the main features of the interaction described above and isolate the interaction from other factors that may have an impact on it. To do this, we use the term, location, as an abstract and symbolic representation of a country, a company, etc. Furthermore, instead of modeling the real-life policy-making process in institutions (e.g. government or board), we assume that there is a social planner in each location who can determine policies alone. Meanwhile, the interactions among policy-takers are simplified as a coordination game, which involves a trade-off between efficiency and risk and has two strict Nash equilibria. The specific features of this simple game make it possible to explore the effect of policies on the selection of social conventions. Hence our model builds on the literature studying the equilibrium selection through learning dynamics (see, for example, Kandori, Mailath, and Rob, 1993; Ellison, 1993; Robson and Vega-Redondo, 1996; Alós-Ferrer and Weidenholzer, 2007, 2008).

In the model there are two stages and two locations. Stage 1 is a static game among rational social planners, who set capacity and mobility constraints for the respective locations to optimize certain objective functions. Stage 2 consists of a learning dynamics, where boundedly rational players, taking the policies in both locations as given, are randomly matched within each location to play the coordination game, and relocate if possible. The structure of the model is related to Alós-Ferrer, Kirchsteiger, and Walzl (2010). They build a two-stage model, in which rational market platform designers try to maximize the profits of their platforms, and the boundedly rational traders choose among the platforms by imitating the most successful behavior in the previous period. As in that model, we will use an approach similar to "backward induction". That is, we first identify the long-run equilibria (LRE) in stage 2 for all possible policy profiles, and then include them into the social planners' optimization problem in Stage 1.

The dynamics in Stage 2 encompasses the multiple location models[1] of Ely (2002) and Anwar (2002). Ely (2002) suggests that if all players can freely move and there are no capacity constraints for both locations, myopic best reply would lead all players to coordinate on the Pareto-efficient equilibrium in the long run. Anwar (2002) considers symmetric restrictions on both the capacity of locations and the mobility of players, and shows that if the effective capacity of each location is relatively large, the states presenting co-existence of conventions (i.e. players coordinate on different equilibria in different locations) are the LRE; however, if the effective capacity is small, the risk-dominant equilibrium will prevail in both locations. We include the results of both models

[1] Other location models include Dieckmann (1999), Blume and Temzelides (2003), and Bhaskar and Vega-Redondo (2004); see Weidenholzer (2010) for a review.

above as particular cases, and identify the LRE when policies in both location are different. Roughly speaking, if the effective capacities in both locations are relatively small, the states with global coordination on the risk-dominant equilibrium are the LRE. If at least one of the locations has relatively large effective capacity, the co-existence of conventions will resut, where the location with smaller effective capacity coordinates on the Pareto-efficient equilibrium.

We consider two different social welfare functions in stage 1. In the first one, the social planners are only concerned with efficiency, i.e. individual average payoffs. We find that there is a set of Nash equilibria (NE) in the game among social planners, which lead to either the global coordination on the risk-dominant equilibrium, or the co-existence of conventions. However, neither the symmetric constraint setting, which leads to the co-existence of conventions in Anwar (2002), nor the unconstrained setting, which leads to the globally efficient outcome in Ely (2002), is a NE. In the second social welfare function, the social planners are also concerned with scale, i.e. total location payoffs. Then, both planners will completely forbid migration. The two isolated locations will then end up coordinating on the risk-dominant equilibrium.

The paper is organized as follows. Section 2 presents the learning dynamics in Stage 2. Section 3 revisits Anwar (2002). We find some incorrections in that work and show that, contrary to the main statement there, global coordination on the Pareto-efficient equilibrium can be selected in the long run. Furthermore, if the payoff of the risk-dominant equilibrium is large enough, the co-existence of conventions occurs for a smaller set of parameters than claimed in Anwar (2002). Section 4 analyzes the general model with active policy-making and presents the results for both social welfare functions. Section 5 concludes. All proofs are relegated to the Appendices.

2 The Learning Model

2.1 Model Setup

This section introduces the learning dynamics in Stage 2. It is closely related to Anwar (2002), and includes Ely (2002) as a particular case. Suppose that there are a total of $2N$ individuals distributed in two locations. Each location $k \in \{1, 2\}$ is characterized by a pair of parameters (p_k, c_k) with $0 \leq p_k \leq 1$ and $1 \leq c_k \leq 2$,[2] such that $\lceil p_k N \rceil$ is the number of immobile individuals and $\lfloor c_k N \rfloor$ is the capacity of location k.[3] Then, the *effective* maximum capacity in location k is $M_k \equiv \lfloor d_k N \rfloor$ and the minimum number of players in location k is $m_k \equiv 2N - M_\ell = 2N - \lfloor d_\ell N \rfloor$, with $d_k = \min\{c_k, 2 - p_\ell\}$, for $k, \ell \in \{1, 2\}, k \neq \ell$. That is, a location has the maximum number of residents either by reaching its capacity constraints, or by having all individuals, apart from the immobile ones in the other location, accommodated there. Here, (p_k, c_k) can be regarded as policies of location $k = 1, 2$. When we focus on the learning dynamics, they are treated as exogenous parameters. However, when active policy-making is involved in the general model (in Section 4), they are the strategies of social planners.

	P	R
P	e, e	g, h
R	h, g	r, r

Table 1: The basic coordination game

Time is discrete, i.e. $t = 1, 2, \ldots$. In each period t, individuals within each location interact with each other by playing the coordination game in Table 1, where[4] $e, r, g, h > 0$, $e > h$, $r > g$,

[2]Unlike in Anwar (2002), here the boundary values of c_k and p_k are included in the model, hence incorporating Ely (2002) as a particular case.

[3]We denote by $\lceil x \rceil$ the minimum integer that is weakly larger than x, and by $\lfloor x \rfloor$ the maximum integer that is weakly smaller than x.

[4]In Anwar (2002), e, g, h and r are not required to be larger than zero. We add this assumption here, to avoid negative payoffs in the social planner's game later. It has no effect on the LRE of the learning dynamics.

$e > r$ and $h + r > e + g$. Hence, (P, P) is the Pareto-efficient equilibrium and (R, R) is the risk-dominant equilibrium. Let $\Pi : S \times S \to \mathbb{R}^+$ be the payoff function of the game, where $S = \{P, R\}$ is the strategy set of each player. $\Pi(s_i, s_j)$ denotes the payoff of playing s_i against s_j. We denote by q^* the probability of playing P in the mixed-strategy NE, i.e. $q^* = \frac{r-g}{e-g-h+r}$. Since (R, R) is risk-dominant, q^* is strictly larger than 1/2. To facilitate the discussion later, we also define $\hat{q}$ through the equality $\Pi(P, (\hat{q}, 1 - \hat{q})) = r$, which yields $\hat{q} = \frac{r-g}{e-g}$. That is, $\hat{q}$ is the probability of playing P in a mixed-strategy such that playing P against it gives the same payoff as in the risk-dominant equilibrium.

Every period, each individual can adjust his strategy. Further, each mobile individual has an opportunity to relocate with an independent and identical positive probability. However, even those who have an opportunity to relocate may not be allowed to reside in another location because of the capacity constraint. Let $\lfloor d_k N \rfloor - n_k$ be the number of vacancies in location k, where n_k is the current population. If the number of players who can move to location k is larger than this number, then only $\lfloor d_k N \rfloor - n_k$ of them will actually be able to move.

Let q_k^t be the proportion of P-players in location $k \in \{1, 2\}$ at period t. We assume that, in each period $t + 1$, each player observes q_k^t for both $k = 1, 2$, and computes the payoff of playing $s \in \{P, R\}$ in location k, $\bar{\Pi}^{t+1}(s, k)$, based on the strategy distribution of players in location k at period t; that is,

$$\bar{\Pi}^{t+1}(s, k) \equiv \Pi(s, (q_k^t, 1 - q_k^t)) = q_k^t \Pi(s, P) + (1 - q_k^t)\Pi(s, R). \tag{1}$$

Then, each player will choose a strategy that maximizes $\bar{\Pi}^{t+1}(s, k)$, given the capacity and mobility constraints.

More specifically, an individual who cannot relocate will choose a strategy $s_i^{t+1} \in \{P, R\}$ in period $t + 1$ given his current location, such that

$$s_i^{t+1} \in \arg\max_{s_i'} \bar{\Pi}^{t+1}(s_i', k_i^t) \tag{2}$$

where k_i^t denotes individual i's location in period t. An individual who can relocate will choose a strategy s_i^{t+1} and a location k_i^{t+1}, such that

$$(s_i^{t+1}, k_i^{t+1}) \in \arg\max_{(s_i', k_i')} \bar{\Pi}^{t+1}(s_i', k_i'). \tag{3}$$

If several choices give a player maximum payoff, he will play each of them with a positive probability.

There are two alternative interpretations for the payoff function $\bar{\Pi}^{t+1}$. The first is that, as assumed in Anwar (2002) and Ely (2002), players are randomly matched within each location in each period to play the coordination game. The behavior updating rule described above corresponds to myopic best reply to the state; that is, players maximize the expected payoff given the strategy distribution of each location in the last period, subject to capacity and mobility constraints. When computing the best response to the state in the other location, this approach is equivalent to standard myopic best reply; that is, a player will choose a strategy to maximize the expected payoff, given the strategies played by all the remaining players in the last period. However, when computing the best response to the state in the current location, this approach differ from standard myopic best reply, in that the player counts himself in the last period as an opponent. For large populations, this corresponds to Oechssler's (1997) best response for large population. That is, if the population is large enough, it is innocuous for an individual to include himself in the strategy profile of the opponents when he computes his expected payoff, since the strategy of one player almost has no effect on the strategy distribution of large population. For small population, though, this kind of behavior can differ from myopic best reply, as the individual ignores the fact that changing his strategy also changes the population proportions at his location.

The second interpretation is that players repeatedly interact with all the other players within the same location (round-robin tournament), as in Kandori, Mailath, and Rob (1993) and Alós-Ferrer (2008). The behavior updating rule above can be interpreted as maximizing the average payoff given the strategy distribution in each location in the last period, while each player includes himself in the strategy profile of the opponents when computing the average payoff in the current location.

2.2 Absorbing Sets

Let $\omega = (v_1, v_2, n_1)$ represent a state of the dynamics described above, where v_k denotes the number of P-players in location k, for $k = 1, 2$, and n_k denotes the total number of players in location k. Hence, the state space is

$$\Omega = \{(v_1, v_2, n_1) | v_1 \in \{0, 1, \ldots, n_1\}, v_2 \in \{0, 1, \ldots, 2N - n_1\}, n_1 \in \{2N - \lfloor d_2 N \rfloor, \ldots, \lfloor d_1 N \rfloor\}\}.$$

Denote

$$q_1(\omega) = \begin{cases} \frac{v_1}{n_1}, & \text{if } n_1 > 0 \\ 0, & \text{otherwise} \end{cases} \quad \text{and} \quad q_2(\omega) = \begin{cases} \frac{v_2}{2N - n_1}, & \text{if } n_1 < 2N \\ 0, & \text{otherwise} \end{cases} \tag{4}$$

We can equivalently rewrite a state as $\omega = (q_1, q_2, n_1)$. The stochastic dynamics above gives rise to a Markov process, whose transition matrix is given by $P = [P(\omega, \omega')]_{\omega, \omega' \in \Omega}$. An absorbing set is a minimal subset of states such that, once the process reaches it, it never leaves.

Lemma 1. *The absorbing sets of the unperturbed process above depend on d_k for $k = 1, 2$ as follows.*

(a) If $d_k < 2$ for both $k = 1, 2$, there are four absorbing sets: $\Omega(PR)$, $\Omega(RP)$, $\Omega(RR)$ and $\Omega(PP)$.

(b) If $d_1 = 2$ and $d_2 < 2$, there are three absorbing sets: $\Omega(RO)$, $\Omega(PO)$, and $\Omega(RP)$. Similarly, if $d_2 = 2$ and $d_1 < 2$, the absorbing sets are: $\Omega(OR)$, $\Omega(OP)$ and $\Omega(PR)$.

(c) If $d_1 = d_2 = 2$, there are four absorbing sets: $\Omega(RO)$, $\Omega(OR)$, $\Omega(PO)$ and $\Omega(OP)$.

where $\Omega(PR) = \{(1, 0, M_1)\}$, $\Omega(RP) = \{(0, 1, m_1)\}$, $\Omega(RR) = \{(0, 0, n_1) |\ n_1 \in \{m_1, \ldots, M_1\}\}$, $\Omega(PP) = \{(1, 1, n_1) | n_1 \in \{m_1, \ldots, M_1\}\}$, $\Omega(RO) = \{(0, 0, 2N)\}$, $\Omega(PO) = \{(1, 0, 2N)\}$, $\Omega(OR) = \{(0, 0, 0)\}$, and $\Omega(OP) = \{(0, 1, 0)\}$.

Following a standard approach, we introduce mutations in this Markov dynamics. We assume that, in each period, with an independent and identical probability ε, each agent randomly chooses a strategy and a location if possible. The perturbed Markov dynamics is ergodic, i.e. there is a unique invariant distribution $\mu(\varepsilon)$. We want to consider small perturbations. It is a well-established result that $\mu^* = \lim_{\varepsilon \to 0} \mu(\varepsilon)$ exists and is an invariant distribution of the unperturbed process P. It describes the time average spent in each state when the original dynamics is slightly perturbed and time goes to infinity. The states in its support, $\{\omega | \mu^*(\omega) > 0\}$, are called *stochastically stable states* or long-run equilibria (LRE).

The standard approach to finding the LRE in the literature of learning in games relies on the graph-theoretical techniques developed by Freidlin and Wentzell (1988) and applied by Foster and Young (1990), Kandori, Mailath, and Rob (1993), Vega-Redondo (1997), etc. Here we briefly summarize the approach in order to clarify the notation and results we rely on.

Denote by $\text{Abs}\Omega$ the collection of absorbing sets. An $\tilde{\Omega}$-tree, h, for any absorbing set $\tilde{\Omega} \in \text{Abs}\Omega$ defines a set of ordered pairs $(\Omega' \to \Omega''), \Omega', \Omega'' \in \text{Abs}\Omega$, such that: (1) each absorbing set $\Omega' \in \text{Abs}\Omega / \{\tilde{\Omega}\}$ is the first element of only one pair; (2) for each $\Omega' \in \text{Abs}\Omega / \{\tilde{\Omega}\}$, there exists a path connecting Ω' with $\tilde{\Omega}$. The set of all $\tilde{\Omega}$-trees is denoted as $H_{\tilde{\Omega}}$.

Let $\Omega', \Omega'' \in \text{Abs}\Omega$. Denote by $c(\Omega', \Omega'')$ the minimum number of mutations required for the transition from Ω' to Ω''. The minimum cost of an $\tilde{\Omega}$-tree, h, is denoted as

$$C(h) = \sum_{(\Omega' \to \Omega'') \in h} c(\Omega', \Omega''). \tag{5}$$

Further, denote by $C(\tilde{\Omega})$ the cost of absorbing set $\tilde{\Omega}$; that is, the minimum cost among all the trees with root $\tilde{\Omega}$:

$$C(\tilde{\Omega}) = \min_{h \in H_{\tilde{\Omega}}} C(h). \tag{6}$$

It is a well-established result (see Kandori and Rob, 1995, for details) that the stochastically stable states ω^* are the elements in Ω^* such that

$$\Omega^* \in \arg \min_{\tilde{\Omega} \in \mathrm{Abs}\Omega} C(\tilde{\Omega}) \tag{7}$$

That is, the elements in the absorbing set whose cost is the lowest among all the absorbing sets are the LRE.

The reason for considering absorbing sets only is simple. The states which are excluded from all absorbing sets are called transient. They represent the states that, with positive probability, the dynamics will never move back to. Hence, by definition, for every transient state there exists a mutation-free transition to some absorbing set. However, every transition from an absorbing set to any transient state requires at least one mutation.

For small population size, the analysis of LRE runs into integer problems, which arise because both the number of players in each location and the number of mutants involved in any considered transition need to be integers. For this reason, the boundaries of the sets in the (d_1, d_2)-space in which different LRE are selected are highly irregular. In the limit, these boundaries become clear-cut, but, for a fixed population size, there is always a small area where long-run outcomes are not clear. However, for large population size these boundary areas effectively vanish. In order to tackle this difficulty, we introduce the following definition.

Definition 1. *$\tilde{\Omega} \subseteq Abs\ \Omega$ is the LRE set for large N uniformly for the set of conditions $\{J_z(d_1, d_2) > 0\}_{z=1}^{Z}$ if, for any $\eta > 0$, there exists an integer N_η such that for all $N > N_\eta$, $\tilde{\Omega}$ is the set of LRE for all $(d_1, d_2) \in [0,1]^2$ such that $\{J_z(d_1, d_2) > \eta\}_{z=1}^{Z}$.*

In words, an LRE set for large population size is such that, given "ideal" boundaries described by certain inequalities (which will be specified in each of our results below), for every point arbitrarily close to the boundaries, there exists a minimal population size such that the LRE corresponds to the given set. Further, the definition incorporates a uniform convergence requirement, in the sense that the minimal population size depends only on the distance to the ideal boundaries, and not on the individual point. This uniform convergence property is important in order to be able to analyze the game among social planners in stage 1.

3 The symmetric case with exogenous policies: Anwar (2002) revisited

Before identifying the LRE of the learning dynamics described in Section 2, this section revisits Anwar's (2002) model to solve two problems involved in the analysis there. The first one refers to the integer problem mentioned above. The second problem is more conceptual. For the transition between two specific absorbing sets, it is possible to find another transition procedure which requires less mutants than the one considered in Anwar (2002). We solve these problems and provide the complete result.

3.1 The Corrected Results

The model in Anwar (2002) is a particular case of the model described in Section 2, where $d_1 = d_2 = d$, $0 < p_k < 1$ and $1 < c_k < 2$ for $k = 1, 2$. Hence, the *effective* maximum capacity in each location is $M \equiv \lfloor dN \rfloor$ and the minimum number of players in each location is $m \equiv 2N - M = 2N - \lfloor dN \rfloor$, with $d = \min\{c, 2-p\}$. Additionally, mutations occur in such a way that with probability ε each player randomly chooses a strategy within the current location. That is, mutants only randomize strategies, not locations. However, according to Remark 1, in this specific case, both forms of mutations will lead to the same result, since the most efficient means of mutations is always to change strategies only.

Lemma 2. *The unperturbed dynamics has four absorbing sets. They are:* $\Omega(PR) = \{(1,0,\ M)\}$, $\Omega(RP) = \{(0,1,m)\}$, $\Omega(RR) = \{(0,0,n_1)|n_1 \in \{m,\ldots,M\}\}$ *and* $\Omega(PP) = \{(1,1,\ n_1)|n_1 \in \{m,\ldots,M\}\}$.

To identify the LRE, we use the approach introduced in Section 2. Anwar (2002, Lemma 1) shows that, because the location model is symmetric, when constructing minimum-cost transition trees, it is enough to consider only three absorbing sets, ignoring either $\Omega(PR)$ or $\Omega(RP)$. If $\Omega(PR)$ is the LRE, so is $\Omega(RP)$. Without loss of generality, we ignore $\Omega(RP)$ from now on. We first consider the case where $M = 2N-1$, and hence $m = 1$. To achieve the transition from $\Omega(PR)$ to $\Omega(PP)$, only one mutant is required, which is the same as that required for the reverse transition. Hence, if $\Omega(PR)$ is selected in the lung run, so is $\Omega(PP)$.

Proposition 1. *If* $M = 2N-1$, *the LRE are the elements in* $\Omega(PP)$, $\Omega(PR)$ *and* $\Omega(RP)$.

Proposition 1 says that, if the capacity of each location is large enough, so that all residents but one can be accommodated, then, efficient coordination will prevail globally in the long run, which contradicts the prediction in Anwar (2002).

Then, we consider the case where $M < 2N-1$. Appendix II shows the minimum-cost transitions. A remarkable finding is that there are two possible minimum-cost transition procedures from $\Omega(PR)$ to $\Omega(RR)$. The first transition procedure (TP1) is to have

$$c_1 = \lceil M(1-q^*)\rceil \tag{8}$$

simultaneous mutants in location 1, so that the best reply for players in location 1 is to play R. The basic idea of the second transition procedure (TP2) is to first move as many as possible P-players from location 1 to location 2 and let them play R, and then change the strategy of the remaining P-players in location 1, if any. We find that, if $q^* \geq \hat{q}$, TP2 cannot give rise to less mutants than TP1. However, if $q^* < \hat{q}$, the transition cost through TP2,

$$c_2 = \lceil M(1-\hat{q})\rceil + \lceil m(1-q^*)\rceil, \tag{9}$$

may be less than that of TP1.

Hence, we discuss two cases. We first consider the case where $h \geq r$. Because $q^* \geq \hat{q}$, $\Omega(PR) \to \Omega(RR)$ must have the minimum cost through TP1. Then we consider the case where $h < r$. Since $q^* < \hat{q}$, the minimum-cost transition of $\Omega(PR) \to \Omega(RR)$ depends on the relationship between c_1 and c_2. We are going to show that, the elements in $\Omega(PP)$ are the LRE within certain parameter configurations in both cases. When N is large enough, our result for case 1 is consistent with Anwar (2002). In case 2, we obtain a different prediction. If the payoff of the risk-dominant equilibrium is large enough, even for a large enough N, $\Omega(RR)$ is the LRE in a larger parameter region than that claimed in Anwar (2002).

3.1.1 Case 1: $h \geq r$

Based on the analysis in Appendix II, if $h \geq r$ and $M < 2N-1$, the minimum-cost transition tree of each absorbing set is the same as that in Anwar (2002). Particularly, TP1 leads to the minimum-cost transition from $\Omega(PR)$ to $\Omega(RR)$, simply because $q^* > \hat{q}$. However, the costs may be different from those in Anwar (2002), because the number of the players in each location and the number of mutants required for each transition have to be integers. For this reason, the minimum cost of the $\Omega(PR)$-tree may be the same as that of the $\Omega(PP)$-tree, which is ignored in Anwar (2002). Table 2 shows the minimum-cost transition tree for each absorbing set.

Using the table above, one can easily derive the conditions supporting the selection of different LRE. In particular, one can immediately see that $C(\Omega(PP))$ is *weakly* larger than $C(\Omega(PR))$. When the two costs are equal, it is possible to select the elements in $\Omega(PP)$ as the LRE.

	$\Omega(RR)$	$\Omega(PR)$	$\Omega(PP)$
minimum-cost tree	$\Omega(RR) \leftarrow \Omega(PR) \leftarrow \Omega(PP)$	$\Omega(RR) \rightarrow \Omega(PR) \leftarrow \Omega(PP)$	$\Omega(RR) \rightarrow \Omega(PR) \rightarrow \Omega(PP)$
$C(\tilde{\Omega})$	$\lceil M(1-q^*)\rceil + \lceil m(1-q^*)\rceil$	$\lceil mq^*\rceil + \lceil m(1-q^*)\rceil$	$2\lceil mq^*\rceil$

Table 2: The minimum-cost transitions for each class

Proposition 2. *Let c_1 be given in (8). For $h \geq r$ and $M < 2N-1$, the elements in $\Omega(PP)$ are LRE if and only if*

$$\lceil m(1-q^*)\rceil = \lceil mq^*\rceil \leq c_1.$$

Example 1. Consider the following coordination game.

	P	R
P	17, 17	0, 10
R	10, 0	8, 8

The global population is $2N = 40$. The probability of playing P in the mixed-strategy Nash equilibrium is $q^* = \frac{8}{15}$. Let $d = 1.75$, then the effective capacity in one location is $dN = 35$. Since $d > 2q^*$, Anwar (2002) predicts that only the elements in $\Omega(PR)$ and $\Omega(RP)$ will be selected in the long run. However, a straightforward computation shows that $C(\Omega(RR)) = 20$ and $C(\Omega(PR)) = C(\Omega(PP)) = 6$. Hence, the elements in $\Omega(PP)$ will also be selected in the long run.

The integer problem above will be smoothed as N becomes large enough. The reason is that when N goes to infinity, $C(\Omega(PR)) = C(\Omega(PP))$ if and only if $d = 2$. Hence, for a large enough N, it is very unlikely that the elements in $\Omega(PP)$ will be selected for $d < 2$. We use Definition 1 and provide the following proposition.

Proposition 3. *Let $h \geq r$.*

(a) $\Omega(RR)$ is the LRE set for large N uniformly for $d < 2q^*$; and

(b) $\Omega(PR) \cup \Omega(RP)$ is the LRE set for large N uniformly for $d > 2q^*$.

3.1.2 Case 2: $h < r$

If $h < r$, then $q^* < \hat{q}$. The minimum-cost transition from $\Omega(PR)$ to $\Omega(RR)$ depends on whether or not $c_1 > (\leq) c_2$. If $c_1 \leq c_2$, we have the same transition cost as in Case 1. If $c_1 > c_2$, $C(\Omega(RR)) = c_2 + \lceil m(1-q^*)\rceil$. The minimum transition costs for the other absorbing sets are still the same. The integer problem is independent of the relationship between h and r. As long as $C(\Omega(PR)) = C(\Omega(PP)) < C(\Omega(RR))$, the elements in $\Omega(PP)$ will be the LRE.

Proposition 4. *Let c_1 and c_2 be given in (8) and (9) respectively. For $h < r$ and $M < 2N-1$, the elements in $\Omega(PP)$ are LRE if and only if*

$$\lceil m(1-q^*)\rceil = \lceil mq^*\rceil \leq \min(c_1, c_2).$$

However, the prediction about the long-run equilibria may change even if we disentangle the integer problem by assuming a large enough N. Since it is possible to have a smaller cost for $\Omega(RR)$-tree, the intuition would suggest that, when $c_2 < c_1$, the elements in $\Omega(RR)$ is more likely to be the LRE.

Proposition 5. *Let $h < r$. Denote $\bar{q} = 1/q^* - 2 + 2q^*$ and $\tilde{d} = \frac{2(2q^*-1)}{2q^*-\hat{q}}$. For $\hat{q} > \bar{q}$,*

(a) $\Omega(RR)$ is the LRE set for large N uniformly for $d < \tilde{d}$; and

(b) $\Omega(PR) \cup \Omega(RP)$ is the LRE set for large N uniformly for $d > \tilde{d}$.

For $\hat{q} \leq \bar{q}$, the result is the same as in Proposition 3.

Note that, if $\hat{q} > \bar{q}$, then $\tilde{d} > 2q^*$. That is, if $\hat{q}$ is relatively large, indicating a high payoff of the risk-dominant equilibrium, coordination on the risk-dominant equilibrium will spread globally in the long run even if $d > 2q^*$, which is in contrast with the prediction in Anwar (2002). We illustrate it with the following example.

Example 2. Consider the following 2×2 coordination game.

	P	R
P	$10, 10$	$0, 4$
R	$4, 0$	$9, 9$

The global population is assumed to be $2N = 200$ and the maximum population in one location is $dN = 125$, i.e. $d = 1.25 > 2q^* = 1.2$. Since $d > 2q^*$, Anwar (2002) predicts that the elements in $\Omega(PR)$ and $\Omega(RP)$ are the LRE. It is easy to see that $\hat{q} = 9/10 > q^* = 0.6$. Because $c_1 = 50 > c_2 = 43$, TP2 leads to the minimum cost for the transition from $\Omega(PR)$ to $\Omega(RR)$. A straightforward computation shows that $C(\Omega(RR)) = 73 < C(\Omega(PR)) = 75$, so the elements in $\Omega(RR)$ will be selected in the long run even though $d > 2q^*$.

3.2 When is the result in Anwar(2002) true?

The analysis above shows that the result in Anwar (2002) is incomplete. First, globally efficient coordination can be selected in the long run, in contrast to the main result of Anwar (2002). Second, if the payoff of the risk-dominant equilibrium is close to that of the Pareto-efficient equilibrium (i.e. $\hat{q}$ is large enough), the elements in $\Omega(RR)$ will be selected in a larger parameter region than that given in Anwar (2002). Based on Propositions 3 and 5, it is straightforward to provide a condition under which the result is consistent with Anwar (2002).

Corollary 1. *Let $h > r$ or $\hat{q} \leq \bar{q}$.*

(a) $\Omega(RR)$ is the LRE set for large N uniformly for $d < d^*$; and

(b) $\Omega(PR) \cup \Omega(RP)$ is the LRE set for large N uniformly for $d > d^*$.

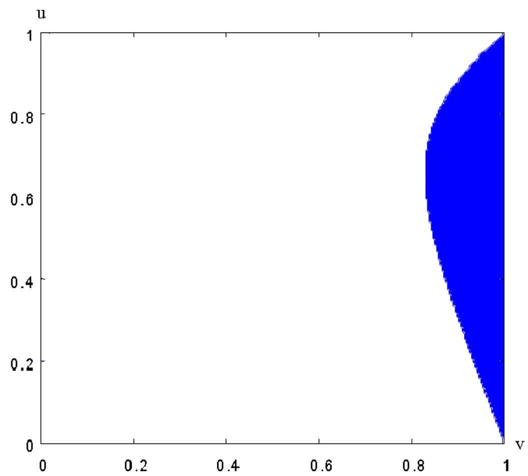

Figure 1: The selection of LRE depends on $\tilde{d}$ in the shadowed area, and d^* in the remaining area.

Let $u = \frac{h-g}{e-g}$ and $v = \frac{r-g}{e-g}$. The coordination game in Table 1 can be normalized as following.

	P	R
P	$1, 1$	$0, u$
R	$u, 0$	v, v

Figure 1 shows the areas in the (u, v)-space, where different cut-off values of d are used to select the LRE. In the shadowed area, the selection of the LRE depends on $\tilde{d}$, hence the predictions of the model do not correspond to the claim in Anwar (2002). In the remaining area, the cut-off value corresponds to d^*. The value of v in the shadowed area is close to 1, implying that the payoff of the risk-dominant equilibrium is close to that of the Pareto-efficient equilibrium, which makes the former one to be the LRE in a larger parameter region than that claimed in Anwar (2002).

4 The general model with endogenous policy-making

This section introduces the general two-stage model, where the social planners set capacity and mobility constraints in stage 1, and the players repeatedly interact with each other within each location in stage 2. We solve the model by backward induction. We first identify the LRE of the learning dynamics in stage 2. The analysis in Section 3 shows that we should pay special attention to the rounding problems and the multiplicity of transition procedures. We assume a large enough N to overcome the first problem (more formally, we rely on the concept introduced in Definition 1, and distinguish cases as before to solve the second one. After that, we investigate the optimal choices of social planners, who can perfectly anticipate the LRE, by considering two different objective functions. We identify the NE in both cases, and show that a slight difference in the objective functions may change the NE of the social planner's game dramatically.

4.1 Model setup

A total of $2N$ individuals are distributed in two different locations $k \in \{1, 2\}$, initially with N players in each location. We assume that each location k has a rational social planner, and refer to the planner in location k as planner k.

There are two stages in the model. Stage 1 is a static game between the two social planners. The planners can neither relocate nor interact with the residents. Instead, to optimize certain objective functions, they will choose a capacity constraint $c_k \in [1, 2]$ and a mobility constraint $p_k \in [0, 1]$ for the location that they are staying in, such that $\lfloor c_k N \rfloor$ determines the maximum capacity and $\lceil p_k N \rceil$ determines the number of immobile players in location k. Stage 2 of the model consists of a discrete-time dynamics described in Section 2, given the policies determined by the social planners in both locations.

4.2 Long-run Equilibria

To identify LRE, we still distinguish two cases, $h \geq r$ and $h < r$, because the minimum-cost transition procedures may differ and that will lead to different predictions.

4.2.1 Case 1. $h \geq r$

We construct minimum-cost transition tree for each absorbing set given in Lemma 1. Note that we cannot ignore either $\Omega(PR)$ or $\Omega(RP)$ as in Section 3. The reason is that social planners can choose different constraints, so that $\Omega(PR)$ and $\Omega(RP)$ are not symmetric.

As an example, we illustrate how to construct the minimum-cost transition trees for the case where $d_k < 2$ for both $k = 1, 2$. There are eight basic transition procedures, which are the transitions between $\Omega(RR)$ and $\Omega(PR)$ ($\Omega(RP)$), and the transitions between $\Omega(PP)$ and $\Omega(PR)$ ($\Omega(RP)$). All these share the common property that the state has to be changed by mutations in only one location. One can immediately see that no minimum-cost transition trees involves a direct transition between $\Omega(PP)$ and $\Omega(RR)$, or between $\Omega(PR)$ and $\Omega(RP)$. These direct transitions require mutations in both locations simultaneously. However, an indirect transition, for example, from $\Omega(PP)$ through $\Omega(PR)$ (or $\Omega(RP)$) to $\Omega(RR)$, only requires mutations in the location with minimum population in each step, hence having a lower cost.

$\Omega(RR) \to \Omega(PR)$	$\Omega(RR) \to \Omega(RP)$	$\Omega(PR) \to \Omega(PP)$	$\Omega(RP) \to \Omega(PP)$
$\lceil m_1 q^* \rceil$	$\lceil m_2 q^* \rceil$	$\lceil m_2 q^* \rceil$	$\lceil m_1 q^* \rceil$
$\Omega(PP) \to \Omega(PR)$	$\Omega(PP) \to \Omega(RP)$	$\Omega(PR) \to \Omega(RR)$	$\Omega(RP) \to \Omega(RR)$
$\lceil m_2(1-q^*) \rceil$	$\lceil m_1(1-q^*) \rceil$	$\lceil M_1(1-q^*) \rceil$	$\lceil M_2(1-q^*) \rceil$

Table 3: The minimum costs for the basic transitions in Case 1a

Appendix II shows that each of these basic transitions has a unique pattern of minimum-cost transition procedure. We summarize them in Table 3. The minimum-cost basic transitions can then be used to construct transition trees for each absorbing set, and compute the minimum cost for each transition tree. The results are shown respectively in Tables 4 and 5 in Appendix III. Then, we compare the minimum costs of different absorbing sets and identify the ones which have the lowest cost.

The analyses for the other cases are similar and provided in Appendix III. We combine all the cases and consider the situation where N is large enough. The condition of a large N smooths the integer problems, and provides clear-cut predictions for the LRE in the main part of the (d_1, d_2)-space.

Theorem 1. *Let $h \geq r$. Denote $\Psi(d_k) = 2 - \frac{1-q^*}{q^*} d_k$ for $k = 1, 2$.*

(a) $\Omega(RR)$ is the LRE set for large N uniformly for $d_1 < \Psi(d_2)$ and $d_2 < \Psi(d_1)$;

(b) $\Omega(RP)$ is the LRE set for large N uniformly for $d_1 > \Psi(d_2)$ and $d_1 > d_2$; and

(c) $\Omega(PR)$ is the LRE set for large N uniformly for $d_2 > \Psi(d_1)$ and $d_1 < d_2$.

Theorem 1 says that, for N large enough, in the main part of the (d_1, d_2)-space, only the elements in three absorbing sets can be selected as stochastically stable. If the effective maximum capacities of both locations are relatively small, the risk-dominant equilibrium will prevail globally. However, if the effective capacity of one location is larger than that of the other, the players in the larger location will coordinate on the risk-dominant equilibrium, while those in the smaller location will coordinate on the Pareto-efficient equilibrium. Figure 2 provides a graphical illustration of Theorem 1. Recalling Definition 1, the theorem says that for any point not on the interior boundaries, there exists a minimal population size such that, for all larger population sizes, the LRE correspond to the given set. For a fixed N, the LRE in the boundary area (illustrated by the area within the dashed lines in Figure 2) may not be clear-cut. In Appendix IV, we show the minimum set of absorbing sets that involves all the possible LRE in different subareas of the boundary area.

4.2.2 Case 2. $h < r$

As analyzed in Appendix II, if $h < r$, the minimum-cost transitions from the absorbing sets with co-existence of conventions to the the absorbing set with global coordination on the risk-dominant equilibrium may be different from those when $h \geq r$.

There are two candidates for minimum-cost transition procedures for each of the transitions mentioned above. We have discussed them in the symmetric case in Section 3. Suppose k is the location where individuals play P. The first candidate is to have a proportion of at least $1 - q^*$ R-mutants in location k, which immediately moves the dynamics to global coordination on R. We refer to this type of transition procedures as TP1. The cost of this transition procedure is

$$c_1^k = \lceil M_k(1 - q^*) \rceil.$$

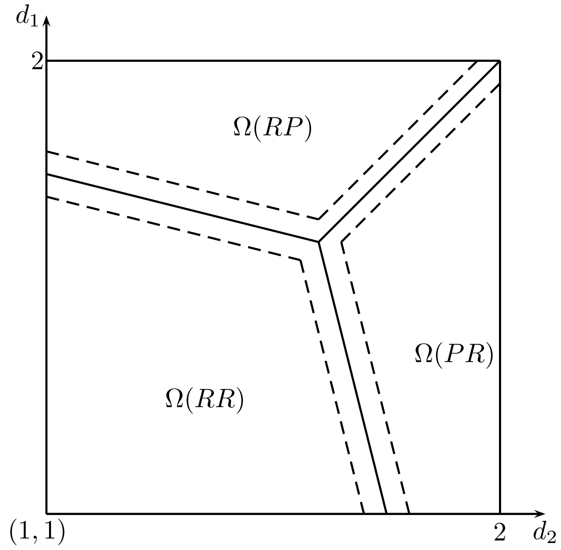

Figure 2: A graphical illustration of the LRE for $h \geq r$.

If $k = 1$, it refers to the cost for the transition $\Omega(PR) \to \Omega(RR)$ through TP1. If $k = 2$, it refers to the cost for $\Omega(RP) \to \Omega(RR)$ through TP1.

Another candidate is to have at least a proportion of $1 - \hat{q}$ players in location k mutate to R, so that myopic best response would suggest that all individuals who have an opportunity to relocate would move to location $\ell \neq k$ and play R, and all the individuals who cannot relocate would remain in location k and play P. Hence, another $\lceil m_k(1 - q^*) \rceil$ R-mutants are required in location k to complete the transition. The cost of this transition procedure, denoted as TP2, is

$$c_2^k = \lceil M_k(1 - \hat{q}) \rceil + \lceil m_k(1 - q^*) \rceil$$

Compare the two costs above for $k = 1, 2$, we have four conditions which divide the (d_1, d_2)-space into four areas. These are the area where $c_1^k > c_2^k$ for both $k = 1, 2$, the areas where $c_1^1 > c_2^1$ and $c_1^2 < c_2^2$ or vice versa, and the area where $c_1^k \leq c_2^k$ for both $k = 1, 2$. Clearly, in the last area, all the results in the case $h \geq r$ hold. In the other three areas, one has to replace c_1^k by c_2^k in all the minimum-cost transition trees whenever $c_1^k > c_2^k$, compare the minimum transition costs for different absorbing sets, and then identify the LRE. We find that the prediction of LRE depends on the value of $\hat{q}$.

Theorem 2. *Let $h < r$. Denote $\Upsilon(d_k) = 2 - \frac{1-\hat{q}}{2q^*-1} d_k$, for $k = 1, 2$. If $\hat{q} > \bar{q}$,*

(a) $\Omega(RR)$ is the LRE set for large N uniformly for $d_1 < \Upsilon(d_2)$ and $d_2 < \Upsilon(d_1)$;

(b) $\Omega(RP)$ is the LRE set for large N uniformly for $d_1 > \Upsilon(d_2)$ and $d_1 > d_2$; and

(c) $\Omega(PR)$ is the LRE set for large N uniformly for $d_2 > \Upsilon(d_1)$ and $d_1 < d_2$.

If $\hat{q} \leq \bar{q}$, the result is the same as stated in Theorem 1.

Theorem 2 says that in the case where $h < r$, if $\hat{q}$ is small, implying that the payoff of the risk-dominant equilibrium is low, the prediction is the same as the case where $h \geq r$. However, if $\hat{q}$ is large, i.e. the payoff of the risk-dominant equilibrium is close to that of the Pareto-efficient equilibrium, then the transition towards $\Omega(RR)$ requires less mutants than in the case where $h \geq r$. Therefore, $\Omega(RR)$ is the LRE in a larger parameter region than in the case $h > r$. The latter result is reflected by the fact that if $\hat{q} > \bar{q}$, then $\Upsilon(d_k) > \Psi(d_k)$ for both $k = 1, 2$, $d_k \in [1, 2]$.

4.3 Nash Equilibria in Stage 1

This subsection identifies the optimal policy chosen by the social planners. The social planners in both locations are rational. They have perfect information about the learning dynamics, and can accurately anticipate the long-run consequences affected by the capacity and mobility constraints of both locations. Hence, given the knowledge of how these constraints will influence the long-run consequences, each planner makes a one-shot decision on these constraints $(c_k, p_k) \in [1,2] \times [0,1]$ of his own location to achieve the particular objective that he is pursuing.

It is often the case that governments or firm managers set policies to achieve specific long-term goals. A government may implement a policy to stimulate the economy in order to, say, achieve a GDP target within a given, fixed time frame. A firm manager may adopt a certain strategy to achieve the goal of, for instance, becoming one of the top three firms in the industry over the course of the next few years. Hence, as in Alós-Ferrer, Kirchsteiger, and Walzl (2010), it is reasonable to focus on the planners' payoffs associated with the limit invariant distribution of the learning dynamics.

We consider two scenarios here. In the first one, the social planners are only concerned with the expected payoff *per person* in their locations. We show that there is a set of Nash equilibria, corresponding to either global coordination on the risk-dominant equilibria or the co-existence of conventions in the long run. In the second scenario, we consider the possibility that the planners also care about scale. In this case, the result is that the planners may completely restrict the mobility of the residents, hence leading the dynamics to a profile where each location has the identical number of players who coordinate on the risk-dominant equilibrium.

4.3.1 Planners only care about efficiency

We first consider the case that the social planners only care about efficiency in the long run. That is, the social planners will maximize the average expected payoff in their respective locations in the LRE. Let $n_k(\omega)$ be the number of individuals in location k in state ω, and $v_k(\omega)$ be the number of P-players in location k in state ω. For any state $\omega = (v_1, v_2, n_1)$, denote the expected/average payoff[5] of P-players and R-players in location k respectively as

$$\begin{aligned}\pi_k(P, v_k(\omega), n_k(\omega)) &= \frac{v_k(\omega)-1}{n_k(\omega)-1}\Pi(P,P) + \frac{n_k(\omega)-v_k(\omega)}{n_k(\omega)-1}\Pi(P,R)\\ \pi_k(R, v_k(\omega), n_k(\omega)) &= \frac{v_k(\omega)}{n_k(\omega)-1}\Pi(R,P) + \frac{n_k(\omega)-v_k(\omega)-1}{n_k(\omega)}\Pi(R,R)\end{aligned}$$

for $n_k(\omega) > 1$. Note that $\pi_k(P, 0, n_k)$ and $\pi_k(R, n_k, n_k)$ are not defined for any n_k. If there is only one player in a location, then the player cannot find a partner to play the game, and we assume that the payoff of this player is zero; that is, $\pi_k(P,1,1) = \pi_k(R,0,1) = 0$.[6]

Then, the average of the expected/average payoff for location k in state ω is

$$\pi_k(\omega) = \frac{v_k(\omega)}{n_k(\omega)}\pi_k(P, v_k(\omega), n_k(\omega)) + \frac{n_k(\omega)-v_k(\omega)}{n_k(\omega)}\pi_k(R, v_k(\omega), n_k(\omega)) \tag{10}$$

for $k = 1, 2$, if $n_k(\omega) > 0$. It is natural to assume that, if a location k is empty, then its social welfare is zero; that is, if $n_k(\omega) = 0$, then $\pi_k(\omega) = 0$.

The limit invariant distribution μ^* is a function of the capacity and mobility constraints. Hence, we have $\mu^*((c_1,p_1),(c_2,p_2)) \in \Sigma(\Omega)$, where $\Sigma(\Omega)$ is the set of probability distributions over Ω. Then, we can define the long-run social welfare function of location $k = 1, 2$ as

$$W_k^E((c_1,p_1),(c_2,p_2)) = \sum_{\omega\in\Omega}\mu^*((c_1,p_1),(c_2,p_2))(\omega)\pi_k(\omega), \tag{11}$$

[5] If players are randomly matched, it refers to the expected payoff. If players have round-robin tournament, it refers to the average payoff.

[6] Alternatively, as in Ely (2002), we can assume that the loner will obtain some positive payoff that is smaller than the payoff the risk-dominant equilibrium. We can also assume that if the loner plays P, his payoff is $\Pi(P,P)$, and if he plays R, his payoff is $\Pi(R,R)$. None of these assumptions will change the results in the following two theorems.

where $\mu^*((c_1, p_1), (c_2, p_2))(\omega)$ is the probability of ω in the limit invariant distribution given (c_1, p_1) and (c_2, p_2).

Theorem 3. *Let the long-run social welfare function be given by (11). Denote* $\tilde{d} = \frac{2(2q^*-1)}{2q^*-\hat{q}}$. *Let* $\Psi(d_k)$ *and* $\Upsilon(d_k)$ *be as in Theorems 1 and 2. If* $h \geq r$ *or* $\hat{q} \leq \bar{q}$,

(a-i) for any (d_1, d_2) *such that* $d_1 < 2q^*$ *or* $d_2 < 2q^*$ *there exists an integer* $\bar{N}$ *such that for all* $N > \bar{N}$, (d_1, d_2) *corresponds to at least one NE, provided that* $d_1 \neq \Psi(d_2)$ *and* $d_2 \neq \Psi(d_1)$;

(a-ii) for any (d_1, d_2) *such that* $d_1 > 2q^*$ *and* $d_2 > 2q^*$ *there exists an integer* $\bar{N}$ *such that for all* $N > \bar{N}$, *there is no NE corresponding to* (d_1, d_2).

If $h < r$ *and* $\hat{q} > \bar{q}$,

(b-i) for any (d_1, d_2) *such that* $d_1 < \tilde{d}$ *or* $d_2 < \tilde{d}$, *there exists an integer* $\bar{N}$ *such that for all* $N > \bar{N}$, (d_1, d_2) *corresponds to at least one NE, provided that* $d_1 \neq \Upsilon(d_2)$ *and* $d_2 \neq \Upsilon(d_1)$.

(b-ii) for any (d_1, d_2) *such that* $d_1 > \tilde{d}$ *and* $d_2 > \tilde{d}$, *there exists an integer* $\bar{N}$ *such that for all* $N > \bar{N}$, *there is no NE corresponding to* (d_1, d_2).

To maximize the average expected payoff in the respective locations, the optimal policies chosen by the social planners, $(c_k, p_k), k = 1, 2$, will project on (d_1, d_2) stated in (a-i) or (b-i) in the theorem so that either the states with the co-existence of conventions or those with global coordination on the risk-dominant equilibrium will be the LRE. In these areas, for each location k, decreasing d_k can never increase the long-run social welfare; however, increasing d_k may increase the social welfare. The reason that the policy profiles projected on these areas are NE is that the planner in location k cannot increase d_k as he wishes, simply because $d_k = \min\{c_k, 2 - p_\ell\}$. As long as $c_k > 2 - p_\ell$, the attempt to increase d_k by increasing c_k becomes ineffective. Hence, one can always find a policy profile so that increasing c_k cannot improve the long-run social welfare.

As to the LRE shaped by the policies, if the effective capacities (d_k) for both locations in a NE of the social planners' game is small, individuals in both locations would coordinate on the risk-dominant equilibrium in the long run. If the effective capacity of one location is small and that of the other location is large, the individuals in the location with a small effective capacity will coordinate on the Pareto-efficient equilibrium, while those in the location with a large effective capacity will coordinate on the risk-dominant equilibrium. The latter result is interesting, since it shows how two social planners with the same objective function in two initially identical locations may choose different strategies and end up with different profiles in each location in the long run. It also provides a novel explanation for a commonly observed phenomenon in everyday life: it is easier to achieve efficient coordination in a small group than in a large one.

Another remarkable finding is that, having large effective capacities in both locations simultaneously is not stable. If such a situation were to occur, the LRE would be either the co-existence of conventions or global coordination on the Pareto-efficient equilibrium. However, the planner in the location with weakly lower social welfare would always have an incentive to decrease the effective capacity of his location in order to have all the players in his location coordinating on the Pareto-efficient equilibrium. The set of (d_1, d_2) profiles which does not correspond to any Nash equilibrium covers two situations: the case there are no capacity and mobility constraints as in Ely (2002); and the case where both locations have large and identical capacity and mobility constraints as in Anwar (2002). The latter case leads to the co-existence of conventions in the long run. Hence, with endogenous capacity and mobility constraints, it is not possible to select global coordination on the Pareto-efficient equilibrium. Additionally, for N large enough, a symmetric setting of capacity and mobility constraints is unstable if it leads to the co-existence of conventions in the long run. Figure 3 illustrates the results in Theorem 3.

4.3.2 The planners care about scale

Now we assume that the planners are concerned with the total expected payoffs of the individuals in their locations. In some situations, scale is an important concern. For example, a country with

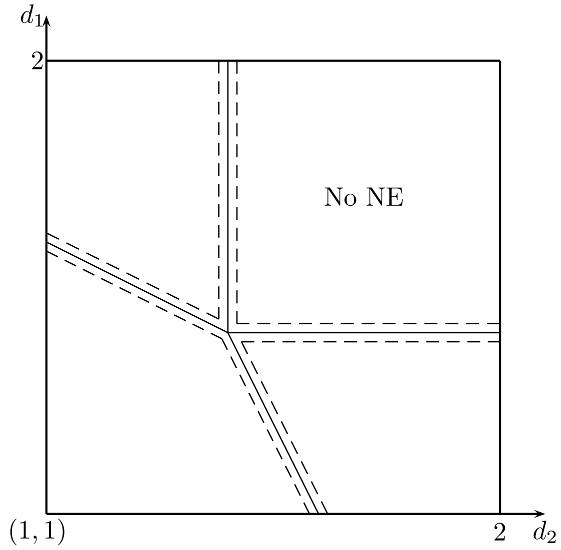

Figure 3: A graphical illustration of Theorem 3 for $h \geq r$ or $\hat{q} \leq \bar{q}$.

a large total GDP attracts more attention and plays a more important role in global economic activities, even if the GDP per capita of this country is still low. A firm with a large scale has more influence on its respective industry, although its performance may be less efficient. To reflect this concern, we consider the following alternative long-run social welfare function for location $k = 1, 2$:

$$W_k^S((c_1, p_1), (c_2, p_2)) = \sum_{\omega \in \Omega} \mu^*((c_1, p_1), (c_2, p_2))(\omega)[\alpha n_k(\omega)\pi_k(\omega) + (1 - \lambda)\pi_k(\omega)], \qquad (12)$$

where $\lambda \in (0, 1]$ models the intensity of planner k's concern with the scale. If $\lambda = 1$, the social planners will only care about the sum of the expected/average payoff of the players in the respective locations. For any $\lambda \neq 1$, the social planners will care about both the sum and the average of the expected/average payoffs for the players in the respective locations.

Theorem 4. *Let the long-run social welfare function be given by (12), and $\tilde{d}$ be as in Theorem 3. Then,*

(a) $d_1 = d_2 = 1$ *corresponds to at least one NE.*

(b) For any $(d_1, d_2) \in [1, 2]^2 \setminus (\{(1, 1)\} \cup \{(2q^*, 2q^*)\})$*, there exists an integer* $\bar{N}$ *such that for all* $N > \bar{N}$*, there is no NE corresponding to* (d_1, d_2)*, if* $h \geq r$ *or* $\hat{q} \leq \bar{q}$*.*

(c) For any $(d_1, d_2) \in [1, 2]^2 \setminus (\{(1, 1)\} \cup \{(\tilde{d}, \tilde{d})\})$*, there exists an integer* $\bar{N}$ *such that for all* $N > \bar{N}$*, there is no NE corresponding to* (d_1, d_2)*, if* $h < r$ *and* $\hat{q} > \bar{q}$*.*

This theorem says that, as long as N is large enough, the only point which will correspond to a NE is $(1, 1)$. The only possible exception may be the point $(2q^*, 2q^*)$ for $h \geq r$ or $\hat{q} \leq \bar{q}$, or $(\tilde{d}, \tilde{d})$ for $h < r$ and $\hat{q} > \bar{q}$. This point might correspond to a NE as well. However, whether or not it corresponds to a NE crucially depends on the parameters of the model, and we cannot provide a general result here.

The intuition for this result is in fact straightforward. For (d_1, d_2) which corresponds to the co-existence of conventions in the long run, the planner in the location with less efficient coordination always has an incentive to decrease the effective capacity of his location to change the LRE to the global coordination on the Pareto-efficient equilibrium. In such a way, the expected population in this location will increase, hence, improving the social welfare. For (d_1, d_2) which leads to the selection of the elements in $\Omega(RR)$ in the long run, being afraid of losing players, the planner in location $k = 1, 2$ always has an incentive to increase the mobility constraint p_k to restrict out-migration. In the end, it reaches a state where no individuals can move out of his current location, and they will coordinate on the risk-dominant equilibrium in both locations in the long run.

Note that the long-run social welfare function given by (11) is a particular case of that given by (12) where $\lambda = 0$. The two theorems above show that for any $\lambda \neq 0$, the NE of the social planners' game is independent of λ. However, when $\lambda = 0$, the result changes dramatically. Hence, the structure of the NE in the social planners' game presents a property of discontinuity as λ converges from one to zero. Put it differently, even if the planners are only slightly concerned with the scale of the locations, this will destabilize the strategy profiles projected on the (d_1, d_2)-space that are proven to be NE in the case where both social planners are only concerned with efficiency, except for those projected on (1,1). Then, the NE will never change no matter how much more weight is put on the sum of the expected/average payoffs.

5 Conclusion

Real-life examples and the literature of learning in games suggest that, in a socio-economic environment, policies interact with social conventions. The behavior of an individual is not only regulated by personal behavioral rules, but also restricted by public policies; in turn, the aggregate behavior of individuals in a society is a foremost concern when social planners design policies to achieve certain objectives. However, the exact mechanism and effect of this interaction has yet to be thoroughly investigated in a formal way. The intention of this paper is to explicitly model rational policy-making in the context of learning in games.

Within a theoretical framework, this task involves endogenizing parameters reflecting policy concerns. Hence, we introduce social planners into a dynamic model of location choice and let them set these parameters. In reality, policy-makers, compared with common individuals, are usually much more far-sighted, have more access to information, and can use information efficiently to achieve their objectives. To capture this fact, we assume that the policy-makers are rational, while the common individuals are boundedly rational, which gives rise to a model of "asymmetric rationality". The policy-makers make decisions first, with perfect knowledge of the effect of different policies on the future of the whole society, and the common individuals take the policy as given and establish social conventions through a learning dynamics. To our knowledge, this is one of the few works that explores the sequential interaction among individuals with different rationality levels involving sequential play within the framework of stochastic learning in games.

Clearly, the objective of social planners has a significant effect on shaping social conventions. In the context of our location model, we investigate two different objective functions. In the first one, the efficiency of coordination in respective locations is the only concern of policy makers. In this case, multiple NE exist, however, a set of symmetric policy arrangements exogenously given in Ely (2002) and Anwar (2002) are not stable. The second objective function is concerned with both scale and efficiency. An interesting finding is that, as long as the policy makers care about scale, even if only a little, this will have the effect that most of the profiles of policy parameters will be unstable. The planners may completely restrict the mobility of the residents and this will lead to the coordination on the less efficient equilibrium in the long run. Hence, our work puts to test the validity of the assumptions of policy constraints that have been considered in the related literature, and also demonstrates how slight policy adjustments may dramatically change the long-run outcomes.

There are many situations in social and economic activities where individuals with different rationality levels interact with each other. Hence, in our opinion, further research should focus on developing more realistic models to analyze such interactions in different contexts. A deeper understanding of these issues will allow us to obtain better insights into the consequences of the interactions among heterogenous individuals. This paper takes a step forward by illustrating that policies interact with social conventions in a nontrivial way, hence, it is necessary to explicitly treat policy parameters and their optimality as important factors for the establishment of social conventions in an organized society.

Appendix I. The absorbing sets of the unperturbed dynamics

Proof of Lemma 1. We show that all the sets listed in lemma 1 are absorbing, i.e. once they are there, the dynamics will remain there forever, and all the other states are transient, i.e. there is a positive probability that the dynamics will never move back to them. Suppose, in period t, the dynamics reaches an arbitrary state ω. Let (k^*, s^*) be a myopic best reply to ω for the individuals who can relocate. Hence, a myopic best reply for the individuals who are currently located in k^* and cannot relocate is s^*. Denote s' a myopic best reply to ω for those who are not in location k^* and cannot relocate. Then, with a positive probability, the individuals in location $\ell^* \neq k^*$ who can relocate will move to location k^* and play s^*, while those residing in location ℓ^* who cannot relocate will play s'. The players in location k^* will remain there and play s^*. Hence, in $t+1$, individuals in location k^* will coordinate on s^*, while those in location ℓ^* will coordinate on s'.

Case I.1: $d_k < 2$ for all $k \in \{1,2\}$. If $s^* = s'$, in $t+2$, all the players will play the same strategy and randomly choose their locations if such an opportunity arises. This corresponds to the set $\Omega(RR)$ or $\Omega(PP)$. They are absorbing, because (R,R) and (P,P) are strict Nash equilibria. Once the dynamics reaches any one of the states in the set $\Omega(RR)(\Omega(PP))$, myopic best reply to the previous state will always lead players to play R (P), and randomize their location choices given the capacity and mobility constraints.

If $s^* \neq s'$, players in one location $k \in \{1,2\}$ must coordinate on P. Then, all the players in the other location who have an opportunity to relocate will move to location k and play P until the population in location k reaches M_k. This corresponds to $\Omega(PR)$ or $\Omega(RP)$. Once there, myopic best reply will lead all the P-players to stay in the current location and play P. The R-players would have an incentive to move to the other location and play P, but are not allowed because of the constraints. Hence, they will play R in their current location.

Case I.2: $d_k = 2$ and $d_\ell < 2$. If $s^* = s'$, with positive probability, all the players will move to location k and play s^*. Once there, the dynamics will remain there forever. This corresponds to $\Omega(PO)$ or $\Omega(RO)$ for $d_1 = 2$, or $\Omega(OP)$ or $\Omega(OR)$ for $d_2 = 2$. If $s^* \neq s'$, individuals will coordinate on P in one location, and R in the other location. If the individuals in location k coordinate on P, while those in location ℓ coordinate on R, all the players in location ℓ will move to k and play P. If the individuals in location k coordinate on R, while those in location ℓ coordinate on P, all the mobile players will move to location ℓ until it reaches its effective maximum capacity. Therefore, the absorbing sets in this case are $\Omega(PO)$, $\Omega(RO)$ and $\Omega(RP)$ for $k = 1$, and $\Omega(OP)$, $\Omega(OR)$ and $\Omega(PR)$ for $k = 2$.

Case I.3: $d_1 = d_2 = 2$. Similarly to Case I.2, if $s^* = s'$, all the players will move to one location and coordinate on s^*. If $s^* \neq s'$, all the players will move to the location with P-players and play P. Once there, the dynamics will stay there forever. Hence, the absorbing sets in this case are $\Omega(RO)$, $\Omega(PO)$, $\Omega(OR)$ and $\Omega(OP)$. □

Proof of Lemma 2. If both locations have identical capacity and mobility constraints, we have $d_1 = d_2 = d$. Clearly, all the corresponding arguments and results in the proof of Lemma 1 hold for this particular case. Since $d < 2$ in Anwar (2002), the conclusion follows. □

Appendix II. The basic minimum-cost transitions among the absorbing sets

In Appendix II, we are going to systematically identify the minimum-cost transitions among all the absorbing sets, and compute the corresponding costs.

Case II.1: $d_k < 2$ for both $k = 1, 2$. There are four absorbing sets $\Omega(PR)$, $\Omega(RP)$, $\Omega(PP)$ and $\Omega(RR)$.

$\Omega(PR)(\Omega(RP)) \to \Omega(PP)$. Consider $\Omega(PR) \to \Omega(PP)$. In the set $\Omega(PR)$, the R-players in location 2 would have an incentive to move to location 1 and play P, but are not allowed to because of the capacity and mobility constraints. Hence, the only way to reach $\Omega(PP)$ is to have enough P-players in location 2 for the share of the P-players to be weakly larger than q^*. Then it is not possible to complete the transition with less than $\lceil m_2 q^* \rceil$ mutants, because m_2 is the minimum population in location 2. Note that in state $\Omega(PR)$ the population in location 2 is exactly m_2. Hence, the minimum cost for this transition is $\lceil m_2 q^* \rceil$. Similarly, the minimum cost for the transition from $\Omega(RP)$ to $\Omega(PP)$ is $\lceil m_1 q^* \rceil$.

$\Omega(PP) \to \Omega(PR)(\Omega(RP))$. The argument is analogous to that for the transitions above. Consider the transition from $\Omega(PP)$ to $\Omega(PR)$. The only possibility to complete the transition is to have enough R-players in location 2 for the share of the R-players to be weakly larger than $1-q^*$. No transition with less than $\lceil m_2(1-q^*) \rceil$ R-mutants can be successful, because the minimum population in location 2 is m_2. In any state in the absorbing set $\Omega(PP)$, $n_2 \in \{m_2, \ldots, M_2\}$. Hence, one can pick the state in $\Omega(PP)$ such that the population in location 2 is exactly m_2, and then having $\lceil m_2(1-q^*) \rceil$ R-mutants in location 2 leads to a successful transition. Similarly, for the transition from $\Omega(PP)$ to $\Omega(RP)$, the minimum cost for the transition is $\lceil m_1(1-q^*) \rceil$.

$\Omega(RR) \to \Omega(PR)(\Omega(RP))$. A similar argument holds here. Consider the transition from $\Omega(RR)$ to $\Omega(PR)$. The only possibility to complete the transition is to have enough P-players in location 1 for the share of the P-players to be weakly larger than q^*. We claim that no transition with less than $\lceil m_1 q^* \rceil$ mutants can be successful, since the minimum population in location 1 is m_1. In any state in the absorbing set $\Omega(RR)$, $n_1 \in \{m_1, \ldots, M_1\}$. Hence, one can pick the state in $\Omega(RR)$ such that the population in location 1 is exactly m_1, then $\lceil m_1 q^* \rceil$ mutants can complete the transition. Similarly, the minimum cost for the transition from $\Omega(RR)$ to $\Omega(RP)$ is $\lceil m_2 q^* \rceil$ mutants.

Note that, if $M_1 = 2N - 1$, then

$$c(\Omega(PP), \Omega(PR)) = c(\Omega(PR), \Omega(PP)) = c(\Omega(RR), \Omega(RP)) = 1.$$

Analogously, if $M_2 = 2N - 1$, then

$$c(\Omega(PP), \Omega(RP)) = c(\Omega(RP), \Omega(PP)) = c(\Omega(RR), \Omega(PR)) = 1.$$

$\Omega(PR)(\Omega(RP)) \to \Omega(RR)$. Consider the transition from $\Omega(PR)$ to $\Omega(RR)$. The analysis is similar for the transition from $\Omega(RP)$ to $\Omega(RR)$. For a successful transition, the population share of the R-players in location 1 has to be weakly larger than $1 - q^*$. Hence, no transition with less than $\lceil m_1(1-q^*) \rceil$ mutants can be successful.

There are two ways to complete the transition. The first transition procedure (TP1) is to have n_P players directly moving from location 1 to 2 and play R, and then have R-mutants in location 1, so that the population share of R-players in this location is weakly larger than $1 - q^*$. This requires in total $n_P + \lceil (M_1 - n_P)(1-q^*) \rceil$ mutants, which is minimized when $n_P = 0$. Hence, the minimum cost for the transition from $\Omega(PR)$ to $\Omega(RR)$ through TP1 is $c(\Omega(PR), \Omega(RR)|TP1) \equiv c_1^1 = \lceil M_1(1-q^*) \rceil$.

The second transition procedure (TP2) is to first move as many players as possible from location 1 to location 2 and let them play R, and then change the strategy of the remaining players in location 1 to R. To achieve the first step, let n_P players moving from location 1 to 2 and play R. Meanwhile, let $\lceil (M_1 - n_p)(1-\hat{q}) \rceil$ mutations occur in location 1, so that the payoff of the P-players in this location is weakly lower than that of R-players in location 2. As a result, all the mobile players in location 1 will move to location 2 and play R. The number of mutations required for this step is $n_P + \lceil (M_1 - n_P)(1-\hat{q}) \rceil$, which is minimized when $n_P = 0$. That is, the minimum number of mutations is $\lceil M_1(1-\hat{q}) \rceil$. Note that, if $h \geq r$, then $q^* \geq \hat{q}$ and $\lceil M_1(1-q^*) \rceil \leq \lceil M_1(1-\hat{q}) \rceil$. Hence, this transition cannot give rise to the minimum cost. Consider $h < r$. After step 1, the population in location 1 is m_1, and all the players play P. Hence, to complete the transition, $\lceil m_1(1-q^*) \rceil$ further R-mutants are required. As a result, the total number of mutants required for the transition following TP2 is $c(\Omega(PR), \Omega(RR)|TP2) \equiv c_2^1 = \lceil M_1(1-\hat{q})\rceil + \lceil m_1(1-q^*)\rceil$. Since $q^* < \hat{q}$, this cost may be the minimum. To summarize, $c(\Omega(PR), \Omega(RR)) = c_1^1$, if $h \geq r$, and $c(\Omega(PR), \Omega(RR)) = \min\{c_1^1, c_2^1\}$, if $h < r$.

We now show that direct transitions from $\Omega(RR)$ to $\Omega(PP)$ cost at least as much as indirect transitions through $\Omega(PR)$ or $\Omega(RP)$. A direct transition requires $\lceil n_1 q^*\rceil + \lceil n_2 q^*\rceil \geq \lceil 2Nq^*\rceil$ mutants. The cost for the transition through, say, $\Omega(PR)$ is $\lceil m_1 q^*\rceil + \lceil m_2 q^*\rceil \leq \lceil 2Nq^*\rceil$, which is smaller than the direct transition. Hence, in any $\Omega(PP)$-tree , $c(\Omega(RR), \Omega(PP)) > c(\Omega(RR), \Omega(PR)) + c(\Omega(PR), \Omega(PP))$. In any $\Omega(PR)$ or $\Omega(RP)$-tree, $c(\Omega(RR), \Omega(PP)) + c(\Omega(PP), \Omega(PR)\ (\Omega(RP))) > c(\Omega(RR), \Omega(PR)\ (\Omega(RP))) + c(\Omega(PP), \Omega(PR)\ (\Omega(RP)))$. Therefore, no minimum-cost transition tree would involve a direct transition from $\Omega(PP)$ to $\Omega(PR)$. A similar argument holds for the reverse transition and the transition between $\Omega(PR)$ and $\Omega(RP)$.

Remark 1. An important finding is that, in case II.1, the minimum-cost transitions share the common characteristic that mutants change only their strategies, not their locations. Hence, in Anwar (2002)'s model where $d_k < 2$ for both $k = 1, 2$, the minimum-cost transitions, under the assumption that mutants randomize their strategies in their current locations, are the same as those under the assumption that mutants randomly choose their strategies and locations.

Case II.2: $d_k = 2$ and $d_\ell < 2, k, \ell \in \{1, 2\}, k \neq \ell$. Consider the case where $k = 1$ (the case for $k = 2$ is symmetric). There are three absorbing sets in this case, $\Omega(RO)$, $\Omega(PO)$ and $\Omega(RP)$.

$\Omega(RO) \to \Omega(RP)$. One mutant is enough for the transition. Let one player move to location 2 and play P, then all the players in location 1 will move to location 2 and play P. Those who cannot relocate will keep playing R in location 1.

$\Omega(RP) \to \Omega(RO)$. To complete the transition, all the players in location 2 have to play R. Hence, this case is the same as the transition from $\Omega(RP)$ to $\Omega(RR)$ in Case II.1. The transition costs are as computed there.

$\Omega(RP) \to \Omega(PO)$. A successful transition requires all the players in location 1 to play P. Hence, analogously to transition $\Omega(RP) \to \Omega(PP)$ in Case II.1, $c(\Omega(RP), \Omega(PO)) = \lceil m_1 q^*\rceil$.

$\Omega(PO) \to \Omega(RP)$. Note that no transition with less than $\lceil m_1(1-q^*)\rceil$ mutants can be successful. To complete the transition with $\lceil m_1(1-q^*)\rceil$ mutants, the population in location 1 has to be at the minimum. However, without additional mutants, the population in location 1 cannot decrease. Hence, $\lceil m_1(1-q^*)\rceil$ mutants are not enough to complete the transition. To reach the minimum population size in location 1, only one mutant is needed. Let one mutant move to location 2 and play P. With positive probability, the dynamics will move to a state in $\Omega(PP)$ where the population in location 1 is m_1. Then, let $\lceil m_1(1-q^*)\rceil$ mutants play R in location 1; the dynamics will move to $\Omega(RP)$. Hence, $c(\Omega(PO), \Omega(RP)) = 1 + \lceil m_1(1-q^*)\rceil$.

$\Omega(PO) \to \Omega(RO)$. For a successful transition, the population share of R players in location 1 has to be weakly larger than $1-q^*$. Let n_P be the number of P-players who move to location 2. Then, the transition requires $n_P + \lceil(2N - n_P)(1-q^*)\rceil$ mutants, which is minimized when $n_P = 0$. Hence, $c(\Omega(PO), \Omega(RO)) = \lceil 2N(1-q^*)\rceil$.

$\Omega(RO) \to \Omega(PO)$. A direct transition requires at least $\lceil 2Nq^*\rceil$ mutants. However, an indirect transition through $\Omega(RP)$ costs less. As argued above, $c(\Omega(RO), \Omega(RP)) + c(\Omega(RP), \Omega(PO)) = 1 + \lceil m_1 q^*\rceil$. It is the minimum, because at least $\lceil m_1 q^*\rceil$ mutants are required for players in location 1 to coordinate on P. Furthermore, at least one mutant is required to decrease the number of players in location 1 to m_1.

Case II.3: $d_k = 2$ for all $k = 1, 2$. There are four absorbing sets, $\Omega(RO)$, $\Omega(PO)$, $\Omega(OR)$ and $\Omega(OP)$.

$\Omega(RO) \to \Omega(OP)(\Omega(OR) \to \Omega(PO))$. One mutant is enough for this transition. Let one player move to location 2 and play P. In the next period, this player will stay in location 2 and play P, and all the players in location 1 will move to location 2 and play P.

$\Omega(RO) \leftrightarrow \Omega(OR)(\Omega(PO) \leftrightarrow \Omega(OP))$. One mutant is enough to complete the transition. Consider $\Omega(RO) \to \Omega(OR)$ first. Let one player move to location 2 and play R. In the next period, with positive probability, all the players in location 1 will move to location 2 and play R, and the single player in location 2 will stay there and play R, hence the dynamics reaches the absorbing set $\Omega(OR)$. The same argument holds for the transition in the reverse direction and the transition between $\Omega(PO)$ and $\Omega(OP)$.

$\Omega(PO) \to \Omega(RO)$($\Omega(OP) \to \Omega(OR)$). As explained in Case II.3, the minimum cost for this transition is $\lceil 2N(1-q^*)\rceil$.

$\Omega(RO) \to \Omega(PO)$ ($\Omega(OR) \to \Omega(OP)$). For a successful transition, the proportion of the P-players in location 1 has to be at least q^*. Hence, a direct transition requires $\lceil 2Nq^*\rceil$ mutants. Consider an indirect transition through $\Omega(OP)$. As shown above, the transitions $\Omega(RO) \to \Omega(OP)$ and $\Omega(OP) \to \Omega(PO)$ require one mutant respectively. Hence, the total cost is two. As long as N is larger than 2, $\lceil 2Nq^*\rceil > 2$. The same arguments hold for the transition from $\Omega(OR)$ to $\Omega(OP)$.

$\Omega(OP) \to \Omega(RO)$ ($\Omega(PO) \to \Omega(OR)$). A direct transition requires $2N$ mutants. Since as long as there is at least one P-player in location 2, all the R-players will be attracted to location 2 and play P. Consider an indirect transition through $\Omega(PO)$. Then the transition $\Omega(OP) \to \Omega(PO)$ requires one mutant, and the transition $\Omega(PO) \to \Omega(RO)$ entails $\lceil 2N(1-q^*)\rceil$ mutants. Hence, the cost for an indirect transition is smaller than that of a direct one.

Appendix III. Proofs for Section 3

We first prove the following technical lemmata to facilitate the remaining proofs. Let h, h' be transition trees rooted on some absorbing sets. Recalling the computations in Appendix II, in our model, for any h, h', the inequality $C(h') \geq C(h)$ can be rewritten as

$$\sum_{\{\alpha'\to\beta'\}\in h} \lceil f^{\alpha'\beta'}(\lfloor d_1N\rfloor, \lfloor d_2N\rfloor)\rceil - \sum_{\{\alpha\to\beta\}\in h} \lceil f^{\alpha\beta}(\lfloor d_1N\rfloor, \lfloor d_2N\rfloor)\rceil \geq 0, \tag{13}$$

where $f^{\alpha\beta}: \mathbb{R}^2_+ \to \mathbb{R}_+$ takes the form $f^{\alpha\beta}(x_1, x_2) = a_1^{\alpha\beta}x_1 + a_2^{\alpha\beta}x_2 + b_1^{\alpha\beta}N + b_2^{\alpha\beta}$ for certain $a_1^{\alpha\beta}, a_2^{\alpha\beta}, b_1^{\alpha\beta}$ and $b_2^{\alpha\beta} \in \mathbb{R}$ for all $\alpha, \beta \in \mathrm{Abs}\Omega$. Further, let

$$F^{hh'}(d_1N, d_2N) \equiv \sum_{\{\alpha'\to\beta'\}} f^{\alpha'\beta'}(d_1N, d_2N) - \sum_{\{\alpha\to\beta\}} f^{\alpha\beta}(d_1N, d_2N) = G^{hh'}(d_1, d_2)N + b_2^{hh'}. \tag{14}$$

Note that $G^{hh'}(d_1, d_2)$ does not depend on N, and $F^{hh'}(d_1N, d_2N)$ would be the same as the left hand side of (13) if all rounding operators could be ignored.

Lemma 3. *Let h, h' be transition trees. If $G^{hh'}(d_1, d_2) > 0$, then there exists an integer $\bar{N}$ such that for all $N > \bar{N}$, $C(h') \geq C(h)$.*

Proof. The inequality (13) is equivalent to

$$G^{hh'}(d_1, d_2)N + b_2^{hh'} + \Delta^{hh'} \geq 0, \tag{15}$$

where

$$\Delta^{hh'} = \sum_{\{\alpha'\to\beta'\}\in h'} \delta^{\alpha'\beta'} - \sum_{\{\alpha\to\beta\}\in h} \delta^{\alpha\beta}, \tag{16}$$

with

$$\delta^{\alpha\beta} = \lceil f^{\alpha\beta}(\lfloor d_1N\rfloor, \lfloor d_2N\rfloor)\rceil - f^{\alpha\beta}(\lfloor d_1N\rfloor, \lfloor d_2N\rfloor) - \sum_{k=1}^{2} a_k^{\alpha\beta}(d_kN - \lfloor d_kN\rfloor) \tag{17}$$

for all $\alpha, \beta \in \mathrm{Abs}\Omega$. Note that in (17), $\lceil f^{\alpha\beta}(\lfloor d_1N\rfloor, \lfloor d_2N\rfloor)\rceil - f^{\alpha\beta}(\lfloor d_1N\rfloor, \lfloor d_2N\rfloor) \in [0, 1[$, $d_kN - \lfloor d_kN\rfloor \in [0, 1[$ and $a_k^{\alpha\beta}$ is fixed for all $\alpha, \beta \in \mathrm{Abs}\Omega$ and $k = 1, 2$. Hence $\delta^{\alpha\beta}$ is a bounded function for all $\alpha, \beta \in \mathrm{Abs}\Omega$. Therefore, according to (16), $\Delta^{hh'}$ is a bounded function, since $\mathrm{Abs}\Omega$ is finite. Further, note that (15) is equivalent to

$$G^{hh'}(d_1, d_2) + \frac{b_2^{hh'}}{N} + \frac{\Delta^{hh'}}{N} \geq 0. \tag{18}$$

Since $\Delta^{hh'}$ is a bounded function, and $b^{hh'}$ is a fixed parameter, when N goes to infinity, $\frac{b_2^{hh'}}{N} + \frac{\Delta^{hh'}}{N}$ will converge to 0. Hence, if $G^{hh'} > 0$ there exists an integer $\bar{N}$ such that for all $N > \bar{N}$ (18) holds, which is equivalent to $C(h') \geq C(h)$. ∎

Lemma 4. *For any $\eta > 0$, there exists an integer N_η such that $G^{hh'}(d_1, d_2) > \eta$ implies $C(h') > C(h)$ for all $N > N_\eta$.*

Proof. The statement follows from (18) and the fact that $\frac{b_2^{hh'}}{N} + \frac{\Delta^{hh'}}{N} \to 0$ as N goes to infinity. $\square$

Proof of Proposition 1. According to the analysis in Case II.1, if $M = 2N - 1$, $c(\Omega(PP), \Omega(PR)) = c(\Omega(PR), \Omega(PP)) = c(\Omega(RR), \Omega(PR)) = 1$. Further, according to Lemma 1 in Anwar (2002), when constructing transition trees in this symmetric case, one can ignore either $\Omega(PR)$ or $\Omega(RP)$. Without loss of generality, we ignore $\Omega(RP)$ here. Consider the following $\Omega(PR)$-tree: $\Omega(RR) \to \Omega(PR) \leftarrow \Omega(PP)$. The cost of this transition tree is 2, which is the lowest possible for a transition tree with three absorbing sets. By changing the direction of the arrow from $\Omega(PP)$ to $\Omega(PR)$ we have a $\Omega(PP)$ tree with the lowest possible cost 2. By the analysis in Appendix II, no minimum-cost transition trees involve the transition between $\Omega(PP)$ and $\Omega(RR)$. Hence, the minimum-cost $\Omega(RR)$-tree must be of the form $\Omega(PP) \to \Omega(PR) \to \Omega(RR)$. Again, according to the analysis in Appendix II, the transition from $\Omega(PP)$ to $\Omega(PR)$ cannot be achieved with 1 mutant only, hence any $\Omega(RR)$-tree has a cost strictly larger than 2. Therefore, the elements in $\Omega(PP)$, $\Omega(PR)$ and $\Omega(RP)$ will be selected in the long run. $\square$

Proof of Proposition 2. The elements in $\Omega(PP)$ will be selected in the long run if and only if $C(\Omega(PP))$ has the minimum cost among all the absorbing sets. The condition follows from Table 2. $\square$

Proof of Proposition 3. As in the proof of Proposition 1, we ignore $\Omega(RP)$. Table 2 shows that $C(\Omega(PP)) \geq C(\Omega(PR))$ always holds, because $q^* > 1/2$. Hence, $\Omega(RR)$ is the unique LRE if and only if $C(\Omega(RR)) < C(\Omega(PR))$. Let h be a minimum-cost $\Omega(RR)$-tree, and h' be a minimum-cost $\Omega(PR)$-tree. Using Table 2, we obtain $F^{hh'}(dN, dN) = 2Nq^* - dN$, hence $G^{hh'}(d, d)$ reduces to $2q^* - d$ (Note that in the symmetric case, $d_1 = d_2 = d$). By Lemma 4, for any $\eta > 0$, there exists an integer N_η such that for all $N > N_\eta$, $2q^* - d > \eta$ implies $C(\Omega(PR)) > C(\Omega(RR))$. By Definition 1, $\Omega(RR)$ is the LRE for large N uniformly for $d < 2q^*$.

Similarly, $\Omega(PR)$ is the unique LRE if (i) $C(\Omega(RR)) > C(\Omega(PR)$ and (ii) $C(\Omega(PP)) > C(\Omega(PR)$. For condition (i), let h be a minimum-cost $\Omega(PR)$-tree, and h' be a minimum-cost $\Omega(RR)$-tree (h and h' is reversed in this condition). Then $F^{hh'}(dN, dN) = dN - 2Nq^*$ and $G^{hh'}(d, d) = d - 2q^*$. By Lemma 4, for any $\eta > 0$, there exists an integer N_η such that for all $N > N_\eta$, $d - 2q^* > \eta$ implies $C(\Omega(RR) > C(\Omega(PR))$. For condition (ii), let h be a minimum-cost $\Omega(PR)$-tree, and h' be a minimum-cost $\Omega(PP)$-tree. Then $F^{hh'}(dN, dN) = (2-d)(2q^* - 1)N$ and $G^{hh'}(d, d) = (2-d)(2q^* - 1)$. By Lemma 4, for any $\eta > 0$, there exists an integer N_η such that for all $N > N_\eta$, $(2-d)(2q^* - 1) > \eta$ implies $C(\Omega(PP)) > C(\Omega(PR))$. By Definition 1, $\Omega(PR)$ and $\Omega(RP)$ is the LRE for large N uniformly for $d - 2q^* > 0$ and $(2-d)(2q^* - 1) > 0$. Since $2q^* - 1 > 0$, the latter inequality is equivalent to $d < 2$. $\square$

Proof of Proposition 4. If $c_1 \leq c_2$, TP1 has the minimum cost for the transition from $\Omega(PR)$ to $\Omega(RR)$. Hence, the elements in $\Omega(PP)$ will be selected if and only if $C(\Omega(PP)) \leq C(\Omega(PP))$ and $C(\Omega(PP)) \leq C(\Omega(RR))$, which implies $\lceil (2N - \lfloor dN \rfloor)(1 - q^*) \rceil = \lceil (2N - \lfloor dN \rfloor) q^* \rceil \leq c_1$. If $c_1 > c_2$, TP2 has the minimum cost for the transition. The same argument indicates $\lceil (2N - \lfloor dN \rfloor)(1 - q^*) \rceil = \lceil (2N - \lfloor dN \rfloor) q^* \rceil \leq c_2$. Combining the two cases gives the result in the statement. $\square$

Proof of Proposition 5. Let $C(\Omega(RR)|TP1)$ be the cost of $\Omega(RR)$ through TP1, and $C(\Omega(RR)$ $|TP2)$ be that of $\Omega(RR)$ through TP2.

$\Omega(RR)$ will be selected if either (i) $C(\Omega(RR)|TP1) \leq C(\Omega(PR))$ or (ii) $C(\Omega(RR)|TP2) \leq C(\Omega(PR))$. For condition (i), as analyzed in Proof of Proposition 3, $G^{hh'}(d, d) = 2q^* - d$. By Lemma 4, for any $\eta > 0$, there exists an integer N_η such that for all $N > N_\eta$, $2q^* - d > \eta$ implies $C(\Omega(RR)|TP1) > C(\Omega(PR))$. For condition (ii), let h be a $\Omega(RR)$-tree with cost $C(\Omega(RR)|TP2)$, and h' be a minimum-cost $\Omega(PR)$-tree. $C(\Omega(PR))$ is given in table 2, and $C(\Omega(PP)|TP2) = \lceil \lfloor dN \rfloor (1 - \hat{q}) \rceil + \lceil (2N - \lfloor dN \rfloor)(1 - q^*) \rceil$. Then, $G^{hh'}(d, d) = 2(2q^* - 1) - d(2q^* - \hat{q})$. Hence, for any $\eta > 0$, there exists an integer N_η such that for all $N > N_\eta$, $C(\Omega(RR)|TP2) < C(\Omega(PR)$ if

$2(2q^* - 1) - d(2q^* - \hat{q}) > \eta$. Denote $\tilde{d} \equiv \frac{2(2q^*-1)}{2q^*-\hat{q}}$. $2q^* \geq \tilde{d}$ if and only if $\hat{q} \leq 1/q^* - 2 + 2q^* \equiv \bar{q}$. Hence, by Definition 1, for $\hat{q} \leq \bar{q}$, $\Omega(RR)$ is the LRE for large N uniformly for $d < 2q^*$; for $\hat{q} > \bar{q}$, $\Omega(RR)$ is the LRE for large N uniformly for $d < \tilde{d}$.

Similarly, $\Omega(PR)$ will be the unique LRE if (i) $C(\Omega(PR)) < C(\Omega(RR)|TP1)$, (ii) $C(\Omega(PR)) < C(\Omega(RR)\ |TP2)$, and (iii) $C(\Omega(PR)) < C(\Omega(PP))$. We use the same approach as above to derive $G^{hh'}$ for each condition. For condition (i), $G^{hh'}(d,d) = d - 2q^*$. Hence, by Lemma 4, for any $\eta > 0$, there exists an integer N_η such that for all $N > N_\eta$, $C(\Omega(PR)) < C(\Omega(RR)|TP1)$ if $d - 2q^* > \eta$. For condition (ii), $G^{hh'}(d,d) = d(2q^* - \hat{q}) - 2(2q^* - 1)$. By Lemma 4, for any $\eta > 0$, there exists an integer N_η such that for all $N > N_\eta$, $C(\Omega(PR)) < C(\Omega(RR)|TP2)$ if $d - \tilde{d} > \eta$. For condition (iii), as computed in Proof of Proposition 3, for any $\eta > 0$, there exists an integer N_η such that for all $N > N_\eta$, $C(\Omega(PR)) < C(\Omega(PP))$ if $2 - d > \eta$. Again, $2q^* \geq \tilde{d}$ if and only if $\hat{q} \leq \bar{q}$. Hence, by Definition 1, for $\hat{q} < \bar{q}$, $\Omega(PR)$ is the LRE for large N uniformly for $d > 2q^*$ and $d < 2$; for $\hat{q} \geq \bar{q}$, $\Omega(PR)$ is the LRE for large N uniformly for $d > \tilde{d}$ and $d < 2$. Since $\Omega(RP)$-tree is symmetric with $\Omega(PR)$-tree, $\Omega(RP)$ will be selected if the same condition is fulfilled. $\square$

Appendix IV. Proofs for Section 4

Proof of Theorem 1. We apply the same approach in the proof of Proposition 3. Ignoring the rounding operators, we compare $C(h)$ with $C(h')$ to derive $G^{hh'}(d_1, d_2)$ for each pair of h, h'. Then we use Lemma 4 to obtain a sufficient condition for $C(h') > C(h)$.

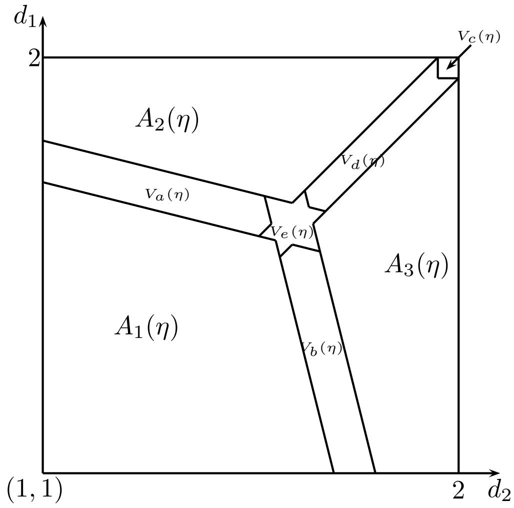

Figure 4: An illustration of the vanishing areas for $h \geq r$

For any $\eta > 0$, denote

$$\begin{aligned}
A_1(\eta) &= \{(d_1, d_2) | \Psi(d_2) - d_1 > \eta, \Psi(d_1) - d_2 > \eta\}; &(19)\\
A_2(\eta) &= \{(d_1, d_2) | d_1 - \Psi(d_2) > \eta, d_1 - d_2 > \eta\}; &(20)\\
A_3(\eta) &= \{(d_1, d_2) | d_2 - \Psi(d_1) > \eta, d_2 - d_1 > \eta\}; &(21)\\
V(\eta) &= [1,2]^2 \setminus (A_1(\eta) \cup A_2(\eta) \cup A_3(\eta)). &(22)
\end{aligned}$$

Note that $V(\eta)$ is vanishing as η decreases: $V(\eta)$ will shrink to the measure-zero set $\{(d_1, d_2)|d_1 = \Psi(d_2) \text{ for } d_2 \in [1, 2q^*]\} \cup \{(d_1, d_2)|d_2 = \Psi(d_1) \text{ for } d_1 \in [1, 2q^*]\} \cup \{(d_1, d_2)|d_1 = d_2 \text{ for } d_2 \in [2q^*, 2]\}$

as $\eta \to 0$. Further, denote

$$\begin{aligned}
V_a(\eta) &= V(\eta) \cap \{(d_1, d_2) | d_1 - d_2 > \eta, \Psi(d_1) - d_2 > \eta\}; \\
V_b(\eta) &= V(\eta) \cap \{(d_1, d_2) | d_2 - d_1 > \eta, \Psi(d_2) - d_1 > \eta\}; \\
V_c(\eta) &= \{(d_1, d_2) | 2 - d_1 \geq \eta, 2 - d_2 \geq \eta\}; \\
V_d(\eta) &= V(\eta) \cap \{(d_1, d_2) | d_1 - \Psi(d_2) > \eta, d_2 - \Psi(d_1) > \eta\} \setminus V_c(\eta); \\
V_e(\eta) &= V(\eta) \setminus (V_a(\eta) \cup V_b(\eta) \cup V_c \cup V_d(\eta)).
\end{aligned}$$

Figure 4 illustrates these subareas.

Case III.1: $d_k < 2$, for both $k = 1, 2$. In this case, there are two singleton absorbing sets and two non-singleton absorbing sets, and, for each of them, there are four candidates for the minimum-cost transition trees. We show the transition trees and their costs in Tables 4 and 5.

	$\Omega(RR)$-tree	$\Omega(PR)$-tree
1	$\Omega(PR) \to \Omega(PP) \to \Omega(RP) \to \Omega(RR)$	$\Omega(RR) \to \Omega(RP) \to \Omega(PP) \to \Omega(PR)$
2	$\Omega(PP) \to \Omega(PR) \to \Omega(RR) \leftarrow \Omega(RP)$	$\Omega(PP) \to \Omega(PR) \leftarrow \Omega(RR) \leftarrow \Omega(RP)$
3	$\Omega(PP) \to \Omega(RP) \to \Omega(RR) \leftarrow \Omega(PR)$	$\Omega(PP) \to \Omega(RP) \to \Omega(RR) \leftarrow \Omega(PR)$
4	$\Omega(RP) \to \Omega(PP) \to \Omega(PR) \to \Omega(RR)$	$\Omega(RP) \to \Omega(PP) \to \Omega(PR) \leftarrow \Omega(RR)$
	$\Omega(RP)$-tree	$\Omega(PP)$-tree
1	$\Omega(PR) \to \Omega(PP) \to \Omega(RP) \leftarrow \Omega(RR)$	$\Omega(RR) \to \Omega(RP) \to \Omega(PP) \leftarrow \Omega(PR)$
2	$\Omega(PP) \to \Omega(PR) \to \Omega(RR) \to \Omega(RP)$	$\Omega(RP) \to \Omega(RR) \to \Omega(PR) \to \Omega(PP)$
3	$\Omega(PR) \to \Omega(RR) \to \Omega(RP) \leftarrow \Omega(PP)$	$\Omega(PR) \to \Omega(RR) \to \Omega(RP) \to \Omega(PP)$
4	$\Omega(RR) \to \Omega(PR) \to \Omega(PP) \to \Omega(RP)$	$\Omega(RR) \to \Omega(PR) \to \Omega(PP) \leftarrow \Omega(RP)$

Table 4: The minimum-cost transition trees in Case III.1.

	$\Omega(RR)$	$\Omega(PR)$	$\Omega(RP)$	$\Omega(PP)$
1	$\lceil m_1(1-q^*) \rceil$ $+\lceil m_2 q^* \rceil$ $+\lceil M_2(1-q^*) \rceil$	$\lceil m_2 q^* \rceil$ $+\lceil m_1 q^* \rceil$ $+\lceil m_2(1-q^*) \rceil$	$2\lceil m_2 q^* \rceil$ $+\lceil m_1(1-q^*) \rceil$	$2\lceil m_2 q^* \rceil$ $+\lceil m_1 q^* \rceil$
2	$\lceil m_2(1-q^*) \rfloor$ $+\lceil M_1(1-q^*) \rceil$ $+\lceil M_2(1-q^*) \rceil$	$\lceil m_2(1-q^*) \rceil$ $+\lceil M_2(1-q^*) \rceil$ $+\lceil m_1 q^* \rceil$	$\lceil m_2 q^* \rceil$ $+\lceil M_1(1-q^*) \rceil$ $+\lceil m_2(1-q^*) \rceil$	$\lceil M_2(1-q^*) \rceil$ $+\lceil m_1 q^* \rceil$ $+\lceil m_2 q^* \rceil$
3	$\lceil m_1(1-q^*) \rceil$ $+\lceil M_1(1-q^*) \rceil$ $+\lceil M_2(1-q^*) \rceil$	$\lceil m_1(1-q^*) \rceil$ $+\lceil M_2(1-q^*) \rceil$ $+\lceil m_1 q^* \rceil$	$\lceil M_1(1-q^*) \rceil$ $+\lceil m_2 q^* \rceil$ $+\lceil m_1(1-q^*) \rceil$	$\lceil M_1(1-q^*) \rceil$ $+\lceil m_2 q^* \rceil$ $+\lceil m_1 q^* \rceil$
4	$\lceil m_2(1-q^*) \rceil$ $+\lceil M_1(1-q^*) \rceil$ $+\lceil m_1 q^* \rceil$	$2\lceil m_1 q^* \rceil$ $+\lceil m_2(1-q^*) \rceil$	$\lceil m_1 q^* \rceil$ $+\lceil m_2 q^* \rceil$ $+\lceil m_1(1-q^*) \rceil$	$2\lceil m_1 q^* \rceil$ $+\lceil m_2 q^* \rceil$

Table 5: The costs for the minimum-cost transition trees in Case III.1.

Let $h_\kappa(\phi)$ be the trees given in Table 4, $\kappa = 1, 2, 3, 4$ and $\phi \in \{RR, PR, RP, PP\}$. For each class of $h_\kappa(\phi)$-trees, one can conduct pairwise comparison for the costs of all candidates and derive $G^{hh'}(d_1, d_2)$ respectively. Omitting the computation details, we list the sufficient condition for each candidate to have minimum cost. For each $\phi \in \Phi$, for any $\eta > 0$, there exists an integer N_η such that for all $N > N_\eta$, (i) $h_1(\phi)$-tree has minimum cost if $(d_1, d_2) \in A_2(\eta) \cap [1, 2[^2$; (ii) $h_2(\phi)$-tree has minimum cost if $(d_1, d_2) \in A_1(\eta) \cap \{(d_1, d_2) | d_1 \geq d_2 + \eta\}$; (iii) $h_3(\phi)$-tree has minimum cost if $(d_1, d_2) \in A_1(\eta) \cap \{(d_1, d_2) | d_1 \leq d_2 - \eta\}$; (iv) $h_4(\phi)$-tree has minimum cost if $(d_1, d_2) \in A_3(\eta) \cap [1, 2[^2$; (v) either $h_1(\phi)$ or $h_2(\phi)$ has the minimum cost if $(d_1, d_2) \in V_a(\eta)$; further, (vi) either $h_2(\phi)$ or $h_3(\phi)$ has the minimum cost if $d_2 - \eta < d_1 < d_2 + \eta$, $d_1 \leq \Psi(d_2) - \eta$ and $d_2 \leq \Psi(d_1) - \eta$; (vii) either $h_3(\phi)$ or $h_4(\phi)$ has the minimum cost if $(d_1, d_2) \in V_b(\eta)$; (viii) either $h_1(\phi)$ or $h_4(\phi)$ has the minimum cost if $(d_1, d_2) \in (V_c(\eta) \cup V_d(\eta))$, (viiii) $h_\kappa(\phi)$ has the minimum cost for either one of $\xi \in \{1, 2, 3, 4\}$ in $V_e(\eta)$.

Fix $\eta > 0$. For each $\kappa \in \{1, 2, 3, 4\}$, we compare the minimum costs among the $h_\kappa(\phi)$-trees for all $\phi \in \Phi$. The elements in the absorbing sets which have the lowest cost among the minimum costs of $h_\kappa(\phi)$-trees will be selected. A straightforward computation shows that, following Definition 1, $\Omega(RP)$ is the LRE for large N uniformly for $d_1 > \Psi(d_2)$ and $d_1 > d_2$; $\Omega(RR)$ is the LRE for large N uniformly for $d_1 < \Psi(d_2)$ and $d_1 < \Psi(d_1)$; $\Omega(PR)$ is the LRE for large N uniformly for $d_2 > \Psi(d_1)$ and $d_1 < d_2$.

Furthermore, in the area $V_a(\eta)$, the LRE set forms a subset of $\Omega(RP) \cup \Omega(RR)$; in the area $V_b(\eta)$, the LRE set forms a subset of $\Omega(RR) \cup \Omega(PR)$; in the area $V_d(\eta)$, the LRE set forms a subset of $\Omega(RP) \cup \Omega(PR)$; and in the area $V_c(\eta)$, the LRE forms a subset of $\Omega(RP) \cup \Omega(PR) \cup \Omega(PP)$.

Case III.2: $d_k = 2$ and $d_\ell < 2$, for $k, \ell = 1, 2$ and $k \neq \ell$. Based on the analysis in Appendix II, we construct the minimum-cost transition trees for all the absorbing sets and show their costs in Table 6.

$k = 1$	$\Omega(RO) \to \Omega(RP) \to \Omega(PO)$	$\Omega(RO) \to \Omega(RP) \leftarrow \Omega(PO)$	$\Omega(PO) \to \Omega(RO) \leftarrow \Omega(RP)$
$C(\cdot)$	$1 + \lceil m_1 q^* \rceil$	$1 + \lceil m_1(1 - q^*) \rceil$	$\lceil 2N(1 - q^*) \rceil + \lceil M_2(1 - q^*) \rceil$
$k = 2$	$\Omega(OR) \to \Omega(PR) \to \Omega(OP)$	$\Omega(OR) \to \Omega(PR) \leftarrow \Omega(OP)$	$\Omega(OP) \to \Omega(OR) \leftarrow \Omega(PR)$
$C(\cdot)$	$1 + \lceil m_2 q^* \rceil$	$1 + \lceil m_2(1 - q^*) \rceil$	$\lceil 2N(1 - q^*) \rceil + \lceil M_1(1 - q^*) \rceil$

Table 6: The minimum-cost transition trees and their costs in Case III.2.

Consider the case where $k = 1$ first. There are three absorbing sets, $\Omega(PO)$, $\Omega(RO)$ and $\Omega(RP)$. Using Table 6, a straightforward comparison shows that $C(\Omega(RO)) > C(\Omega(RP))$ and $C(\Omega(RO)) > C(\Omega(PO))$ if $\lceil 2N(1 - q^*) \rceil > 1$. Hence, if N is large enough, $\Omega(RO)$ can never be selected. Now consider the condition $C(\Omega(RP)) < C(\Omega(PO))$. Let h be the minimum-cost $\Omega(RP)$-tree, and h' be the minimum-cost $\Omega(PO)$-tree. Then, $G^{hh'} = (2 - d_2)(2q^* - 1)$. By Corollary 4, for any $\eta > 0$, there exists an integer N_η such that for all $N > N_\eta$, $C(\Omega(RP)) < C(\Omega(PO))$ if $G^{hh'} = (2 - d_2)(2q^* - 1) \geq \eta$. This condition can be rearranged as $d_2 \leq 2 - \frac{\eta}{2q^* - 1}$. Denoting $\eta' = \frac{\eta}{2q^* - 1}$ and renaming $\eta' = \eta$, we have $d_2 \leq 2 - \eta$. Hence, for any $\eta > 0$, there exists an integer N_η such that for all $N > N_\eta$, $\Omega(RP)$ will be selected if $d_2 \leq 2 - \eta$; the elements in either $\Omega(RP)$ or $\Omega(PO)$ will be selected if $d_2 > 2 - \eta$.

The same argument holds for the case where $k = 2$. Hence, for any $\eta > 0$, there exists an integer N_η, such that for all $N > N_\eta$, $\Omega(PR)$ is the unique LRE if $d_1 \leq 2 - \eta$; the elements in $\Omega(PR)$ or $\Omega(OP)$ will be selected if $d_1 > 2 - \eta$.

Hence, for $d_1 = 2$ and $d_2 < 2$, $\Omega(RP)$ is the LRE for large N uniformly for $d_2 < 2$; for $d_2 = 2$ and $d_1 < 2$, $\Omega(PR)$ is the LRE for large N uniformly for $d_1 < 2$. In area $V_c(\eta)$, the LRE set forms a subset of $\Omega(PR) \cup \Omega(OP) \cup \Omega(PO) \cup \Omega(RP)$.

Case III.3: $d_1 = d_2 = 2$. There are four absorbing sets, $\Omega(RO)$, $\Omega(PO)$, $\Omega(OR)$ and $\Omega(OP)$. Appendix II exhibits that the transitions between $\Omega(PO)$ and $\Omega(OP)$ (in both directions) only need one mutant, just as in the transitions between $\Omega(RO)$ and $\Omega(OR)$. The transition from $\Omega(OR)(\Omega(RO))$ to $\Omega(PO)(\Omega(OP))$ only requires one mutant. Consider the following $\Omega(PO)$-tree.

$$\Omega(RO) \to \Omega(OR) \to \Omega(PO) \leftarrow \Omega(OP)$$

It is easy to see that the minimum cost of this transition tree is 3. For a transition tree with four absorbing sets, it must be the minimum cost. Hence, the element in $\Omega(PO)$ is a LRE. Since the transition from $\Omega(PO)$ to $\Omega(OP)$ needs only one mutant, one can build an $\Omega(OP)$-tree simply by reversing the direction of the arrow between $\Omega(OP)$ and $\Omega(PO)$ in the $\Omega(PO)$-tree above. Therefore, the minimum cost of this $\Omega(OP)$-tree is also 3. Hence, the element in $\Omega(OP)$ is a LRE as well.

Nevertheless, if N is large enough, it is not possible to complete the transition from either $\Omega(PO)$ or $\Omega(OP)$ to either $\Omega(RO)$ or $\Omega(OR)$ by one mutant. Hence, the minimum cost of any

$\Omega(RO)$- or $\Omega(OR)$-tree must be larger than 3. Therefore, for N large enough, neither the element in $\Omega(RO)$ nor that in $\Omega(OR)$ is a LRE.

Consider all the three cases above together. For any $\eta > 0$, there exists an integer N'_η, which is the maximum of all the N_η required in all the cases above, such that when $N > N'_\eta$, all the results above hold simultaneously. By Definition 1, we have the statement in the theorem. □

Corollary 2. *Let $h \geq r$. For any $\eta > 0$, there exists an integer $\bar{N}$, such that for all $N > \bar{N}$, if $(d_1, d_2) \in V(\eta)$, the LRE form a subset of*

(a) $\Omega(RP) \cup \Omega(RR)$ if $(d_1, d_2) \in V_a(\eta)$;

(b) $\Omega(PR) \cup \Omega(RR)$ if $(d_1, d_2) \in V_b(\eta)$;

(c) $\Omega(PR) \cup \Omega(RP) \cup \Omega(OP) \cup \Omega(PO) \cup \Omega(PP)$ if $(d_1, d_2) \in V_c(\eta) \setminus \{(2,2)\}$;

(d) $\Omega(PR) \cup \Omega(RP)$ if $(d_1, d_2) \in V_d(\eta)$; and

(e) $\Omega(PR) \cup \Omega(RP) \cup \Omega(RR)$ if $(d_1, d_2) \in V_e(\eta)$.

(f) Further, the LRE are the elements in $\Omega(PP)$ if $d_1 = d_2 = 2$.

Proof of Corollary 2. The results directly follow the proof of Theorem 1. □

Proof of Theorem 2. If $h < r$, the transition $\Omega(PR) \to \Omega(RR)(\Omega(RO))$ and $\Omega(RP) \to \Omega(RR)$ $(\Omega(OR))$ may have lower costs through TP2 than through TP1. That is, $c_2^k = \lceil M_k(1 - \hat{q}) \rceil + \lceil m_k(1 - q^*) \rceil$ is smaller than $c_1^k = \lceil M_k(1 - q^*) \rceil$. We consider the case where N is large enough. By Corollary 2, for any $\eta > 0$, there exists an integer N_η such that for all $N > N_\eta$, $c_1^k > c_2^k$ if

$$d_k \geq \Lambda(d_\ell) + \eta, \tag{23}$$

and $c_1^k < c_2^k$ if

$$d_k \leq \Lambda(d_\ell) - \eta, \tag{24}$$

where $\Lambda(d_\ell) = 2 - \frac{\hat{q} - q^*}{1 - q^*} d_\ell$ and $\ell \neq k$.

These conditions (for $k = 1, 2$) divide the main area of the (d_1, d_2)-space into four subareas. That is, (23) holds for $k = 1$ and (24) holds for $k = 2$; (23) holds for $k = 2$ and (24) holds for $k = 1$; (23)holds for both $k = 1, 2$; and (24) holds for both $k = 1, 2$. In the last subarea, the transitions through TP2 are always more costly than those through TP1, hence, all the results in the case $h \geq r$ still holds. In the remaining three areas, TP2 leads to the minimum cost for either $\Omega(PR) \to \Omega(RR)(\Omega(RO))$ or $\Omega(RP) \to \Omega(RR)(\Omega(RO))$, or both. We analyze these three cases separately.

We use the same approach as in the proof of Theorem 1. Applying Corollary 4, we first compare the candidate transition trees with the same root, and then identify the absorbing sets which have the lowest cost among all the other absorbing sets. Note that $\Lambda(d_\ell) \geq \Psi(d_\ell)$ for all $d_\ell \in [1, 2]$, $\ell \in \{1, 2\}$ if $\hat{q} \leq \bar{q} \equiv 2q^* - 2 + \frac{1}{q^*}$, and the reverse inequality holds if $\hat{q} > \bar{q}$. Hence, we will distinguish two cases.

Case A. $\hat{q} \leq \bar{q}$. Then $\Lambda(d_\ell) \geq \Psi(d_\ell)$ for all $d_\ell \in [1, 2]$, $\ell \in \{1, 2\}$. First, consider the case where $d_k < 2$ for both $k = 1, 2$. In this case, only $\Omega(PR) \to \Omega(RR)$ and $\Omega(RP) \to \Omega(RR)$ are involved.

Case A.I. (23) holds for $k = 1$ and (24) holds for $k = 2$. In this area, only the transition $\Omega(RP) \to \Omega(RR)$ has a lower cost through TP2 than through TP1. For the transition costs, we can use Table 5 and simply replace c_1^2 by c_2^2. We summarize the results as follows. For any $\eta > 0$, there exists an integer N_η such that for all $N > N_\eta$, (i) among $\Omega(RR)$-trees, $h_1(RR)$ has the minimum cost if $d_1 \geq 2(1 - q^*) + d_2(2q^* - \hat{q}) + \eta$; $h_4(RR)$ has the minimum cost if $d_1 \leq 2(1 - q^*) + d_2(2q^* - \hat{q}) - \eta$. In between, either $h_1(RR)$ or $h_4(RR)$ has the minimum cost. (ii) Among $\Omega(PR)$-trees, $h_1(PR)$ always has the minimum cost in this area. (iii) Among $\Omega(RP)$-trees, $h_1(RP)$ always has the minimum cost. (iv) $C(h_\kappa(PP))$ is always larger than $C(h_\kappa(PR))$ for each $\kappa \in \{1, 2, 3, 4\}$. Comparing the costs of $h_1(RR)$, $h_4(RR)$, $h_1(PR)$ and $h_1(RP)$ in this area, we find that $h_1(RP)$ has the minimum cost, hence the elements in $\Omega(RP)$ are selected in the long run.

In the area $\{(d_1,d_2)|\Lambda(d_2)-\eta < d_1 < \Lambda(d_2)+\eta, d_1 > d_2+\eta\}$, either TP1 or TP2 leads to the minimum-cost transitions. If TP1 leads to the minimum cost, then by the comparisons in the case $h > r$, the elements in $\Omega(RP)$ are selected. If TP2 leads to the minimum cost, using the transition costs in Case A.I, $\Omega(RP)$ is still selected. Hence, combining all the results above, we have that $\Omega(RP)$ is the LRE if $\{(d_1,d_2)|d_1 > \Lambda(d_2)-\eta, d_1 > d_2+\eta\}$.

Case A.II. (23) holds for $k=2$ and (24) holds for $k=1$. We replace c_1^1 by c_2^1 in Table 5 and use the same approach as in case A.I. Symmetrically with Case A.I, we obtain that in the area $\{(d_1,d_2)|d_2 > \Lambda(d_1)-\eta, d_2 \geq d_1+\eta\}$, $\Omega(PR)$ is the LRE.

Case A.III. (23) holds for both $k=1,2$. We replace both c_1^1 by c_2^1 and c_1^2 by c_2^2 in Table 5 and use the same approach as in case A.I. We find that, for any $\eta > 0$, there exists an integer N_η such that for all $N > N_\eta$, (i) $h_1(RR)$ has the minimum cost if $d_1 > d_2+\eta$; $h_4(RR)$ has the minimum cost if $d_1 < d_2-\eta$. (ii) $h_1(PR)$ has the minimum cost if $d_1 > d_2+\eta$; $h_4(PR)$ has the minimum cost if $d_1 < d_2-\eta$. (iii) $h_1(RP)$ has the minimum cost if $d_1 > d_2+\eta$; $h_4(RP)$ has the minimum cost if $d_1 < d_2-\eta$. (iv) $C(h_\kappa(PP))$ is either larger than $C(h_\kappa(RP))$ or larger than $C(h_\kappa(PR))$ for each $\kappa \in \{1,2,3,4\}$. Comparing the costs of the transition trees above, we find that $h_1(RP)$ has the minimum cost if $d_1 > d_2+\eta$; $h_4(PR)$ has the minimum cost if $d_1 < d_2-\eta$.

In the area $\{(d_1,d_2)|\Lambda(d_2)-\eta \leq d_1 \leq \Lambda(d_2)+\eta, d_1 > d_2+\eta\}$, TP2 leads to the minimum cost either only for $\Omega(RP) \to \Omega(RR)$ or for both $\Omega(RP) \to \Omega(RR)$ and $\Omega(PR) \to \Omega(RR)$. If the former is true, using the transition costs in Case A.I, we find that $\Omega(RP)$ is selected. If the latter is true, using the transition costs in case A.III, $\Omega(RP)$ is still selected. Hence, the element in $\Omega(RP)$ is the LRE in the area $\{(d_1,d_2)|d_1 \geq \Lambda(d_2)-\eta, d_1 > d_2+\eta\}$. Symmetrically, the element in $\Omega(PR)$ is the LRE in the area $\{(d_1,d_2)|d_2 \geq \Lambda(d_1)-\eta, d_1 > d_2-\eta\}$.

Case A.IV. (24) holds for both $k=1,2$. In this case, TP2 cannot lead to the minimum cost, hence, the result is the same as in the case $h > r$.

Now consider the case where $d_k = 2$ and $d_\ell < 2$ for $k,\ell = 1,2, k \neq \ell$. Let $k=1$ first. Using Table 6, if TP2 lead to the minimum cost, the cost of $\Omega(RO)$-tree will change to $\lceil 2N(1-q^*)\rceil + \lceil M_2(1-\hat{q})\rceil + \lceil m_2(1-q^*)\rceil$. As long as N is large enough, this cost is still larger than $C(\Omega(RP)) = 1 + \lceil m_1(1-q^*)\rceil$. Hence, the element in $\Omega(RP)$ is still selected. The same argument holds for $k=2$. If N is large enough, the element in $\Omega(PR)$ will still be selected for $k=2$.

Combining all the results above, we obtain the same conclusion as in the case $h > r$.

Case B. $\hat{q} > \bar{q}$. Then $\Lambda(d_\ell) < \Psi(d_\ell)$ for $d_\ell \in [1,2]$ and $\ell \in \{1,2\}$. For any $\eta > 0$, denote

$$\begin{aligned}
B_1(\eta) &= \{(d_1,d_2)|\Upsilon(d_2)-d_1 > \eta, \Upsilon(d_1)-d_2 > \eta\} \\
B_2(\eta) &= \{(d_1,d_2)|d_1-\Upsilon(d_2) > \eta, d_1-d_2 > \eta\} \\
B_3(\eta) &= \{(d_1,d_2)|d_2-\Upsilon(d_1) > \eta, d_2-d_1 > \eta\} \\
U(\eta) &= [1,2]^2 \setminus (B_1(\eta) \cup B_2(\eta) \cup B_3(\eta))
\end{aligned}$$

Similarly to the analysis for the case $h \geq r$, this splits the (d_1,d_2)-space in three main areas. $U(\eta)$ is vanishing as η decreases: it will shrink to the measure-zero set $\{(d_1,d_2)|d_1 = \Upsilon(d_2)$ for $d_2 \in [1,\tilde{d}]\} \cup \{(d_1,d_2)|d_2 = \Upsilon(d_1)$ for $d_1 \in [1,\tilde{d}]\} \cup \{(d_1,d_2)|d_1 = d_2$ for $d_2 \in [\tilde{d},2]\}$ as $\eta \to 0$. Further, denote

$$\begin{aligned}
U_a(\eta) &= Q(\eta) \cap \{(d_1,d_2)|d_1-d_2 > \eta, \Upsilon(d_1)-d_2 > \eta\}; \\
U_b(\eta) &= Q(\eta) \cap \{(d_1,d_2)|d_2-d_1 > \eta, \Upsilon(d_2)-d_1 > \eta\}; \\
U_c(\eta) &= V_c(\eta); \\
U_d(\eta) &= Q(\eta) \cap \{(d_1,d_2)|d_1-\Upsilon(d_2) > \eta, d_2-\Upsilon(d_1) > \eta\} \setminus U_c(\eta); \\
U_e(\eta) &= Q(\eta) \setminus (Q_a(\eta) \cup Q_b(\eta) \cup U_c(\eta) \cup U_d(\eta)).
\end{aligned}$$

Case B.I. (23) holds for $k=1$ and (24) holds for $k=2$. Using the conditions for each candidate transition tree to reach the minimum cost in Case A.I, we obtain the following results. For any $\eta > 0$, there exists an integer N_η such that for all $N > N_\eta$, (i) $h_1(RR)$ has the minimum cost if $d_1 > \Psi(d_2) + \eta$, $h_2(RR)$ has the minimum cost if $d_1 < \Psi(d_2) - \eta$, in between, either $h_1(RR)$ or $h_2(RR)$ has the minimum cost. (ii) $h_1(PR)$ has the minimum cost if $d_1 < \Upsilon(d_2) + \eta$; $h_2(PR)$ has the minimum cost if $d_1 < \Upsilon(d_2) - \eta$; in between, either $h_1(PR)$ or $h_2(PR)$ has the minimum cost. (iii) $h_1(RP)$ has the minimum cost if $d_1 > \Psi(d_2) + \eta$; $h_2(RP)$ has the minimum cost if $d_1 < \Psi(d_2) - \eta$; in between, either $h_1(PR)$ or $h_2(PR)$ has the minimum cost. (iv) $C(h_\kappa(PP))$ is always larger than $C(h_\kappa(PR))$ for each $\kappa \in \{1, 2, 3, 4\}$.

Then we compare the minimum cost transition trees in each of the subareas above to identify the LRE. For any $\eta > 0$, there exists an integer N_η, such that for all $N > N_\eta$, (a) the element in $\Omega(RP)$ will be selected if $d_1 > \Upsilon(d_2) + \eta$. (b) the elements in $\Omega(RR)$ will be selected if $d_1 < \Upsilon(d_2) - \eta$. (c) the element(s) in either $\Omega(RP)$ or $\Omega(RR)$ will be selected if $\Upsilon(d_2) - \eta \leq d_1 \leq \Upsilon(d_2) + \eta$.

Case B.II. (23) holds for $k = 2$ and (24) holds for $k = 1$. Symmetrically, for any $\eta > 0$, there exists an integer N_η, such that for all $N > N_\eta$, $\Omega(RP)$ will be selected if $d_2 > \Upsilon(d_1) + \eta$; the elements in $\Omega(RR)$ will be selected if $d_2 < \Upsilon(d_1) - \eta$; the element(s) in either $\Omega(PR)$ or $\Omega(RR)$ will be selected if $\Upsilon(d_1) - \eta \leq d_2 \leq \Upsilon(d_1) + \eta$.

Case B.III. (23) holds for both $k = 1, 2$. We find that, for any $\eta > 0$, there exists an integer N_η such that for all $N > N_\eta$, (i) $h_1(RR)$ has the minimum cost if $d_1 > d_2 + \eta$ and $d_1 > \tilde{d} + \eta$. $h_2(RR)$ has the minimum cost if $d_1 > d_2 + \eta$ and $d_1 < \tilde{d} - \eta$. If $\tilde{d} - \eta \leq d_1 \leq \tilde{d} + \eta$, either $h_1(RR)$ or $h_2(RR)$ has the minimum cost. $h_3(RR)$ tree has the minimum cost if $d_1 < d_2 - \eta$ and $d_2 < \tilde{d} - \eta$. If $d_2 - \eta \leq d_1 \leq d_2 + \eta$, $d_1 < \tilde{d} - \eta$ and $d_2 \leq \tilde{d} - \eta$, either $h_2(RR)$ or $h_3(RR)$ has the minimum costs. $h_4(RR)$ has the minimum cost if $d_1 < d_2 - \eta$ and $d_2 > \tilde{d} + \eta$. If $\tilde{d} - \eta \leq d_2 \leq \tilde{d} + \eta$ and $d_1 < d_2 - \eta$, either $h_3(RR)$ or $h_4(RR)$ has the minimum cost. (ii) $h_1(PR)$ has the minimum cost if $d_1 > \Upsilon(d_2) + \eta$ and $d_1 > d_2 + \eta$. $h_2(PR)$ has the minimum cost if $d_1 < \Upsilon(d_2) - \eta$ and $d_1 > d_2 + \eta$. If $\Upsilon(d_2) - \eta \leq d_1 \leq \Upsilon(d_1) - \eta$ and $d_1 > d_2 + \eta$, either $h_1(PR)$ or $h_2(PR)$ has the minimum cost. $h_3(PR)$ has the minimum cost if $d_1 < d_2 - \eta$ and $d_2 \leq \tilde{d} - \eta$. If $d_2 - \eta \leq d_1 \leq d_2 + \eta$, $d_1 < \Upsilon(d_2) - \eta$, and $d_2 < \tilde{d} - \eta$, either $h_2(PR)$ or $h_3(PR)$ has the minimum cost. $h_4(PR)$ has the minimum cost if $d_1 < d_2 - \eta$ and $d_2 > \tilde{d} + \eta$. If $\tilde{d} - \eta \leq d_2 \leq \tilde{d} + \eta$ and $d_1 < d_2 - \eta$, either $h_3(PR)$ or $h_4(PR)$ has the minimum cost. (iii) $h_4(RP)$ has the minimum cost if $d_2 > \Upsilon(d_1) + \eta$ and $d_2 > d_1 + \eta$. $h_3(RP)$ has the minimum cost if $d_2 < \Upsilon(d_1) - \eta$ and $d_2 > d_1 + \eta$. If $\Upsilon(d_1) - \eta \leq d_2 \leq \Upsilon(d_2) - \eta$ and $d_2 > d_1 + \eta$, either $h_3(RP)$ or $h_4(RP)$ has the minimum cost. $h_2(PR)$ has the minimum cost if $d_2 < d_1 - \eta$ and $d_1 < \tilde{d} - \eta$. If $d_2 - \eta \leq d_1 \leq d_2 + \eta$, $d_2 < \Upsilon(d_1) - \eta$, and $d_1 < \tilde{d} - \eta$, either $h_2(RP)$ or $h_3(RP)$ has the minimum cost. $h_1(RP)$ has the minimum cost if $d_1 > d_2 + \eta$ and $d_1 > \tilde{d} + \eta$. If $\tilde{d} - \eta \leq d_1 \leq \tilde{d} + \eta$ and $d_1 > d_2 + \eta$, either $h_1(RP)$ or $h_2(RP)$ has the minimum cost. (iv) $C(h_\kappa(PP))$ is either larger than $C(h_\kappa(RP))$ or larger than $C(h_\kappa(PR))$ for each $\kappa \in \{1, 2, 3, 4\}$.

Using the results above and comparing the costs of the trees rooted with different absorbing sets, we obtain the LRE in the different areas. For any $\eta > 0$, there exists an integer N_η such that for all $N > N_\eta$, $\Omega(RR)$ have the minimum cost if $d_1 < \Upsilon(d_2) - \eta$ and $d_2 < \Upsilon(d_1) - \eta$; $\Omega(RP)$ has the minimum cost if $d_1 > \Upsilon(d_2) + \eta$ and $d_1 > d_2 + \eta$. $\Omega(PR)$ has the minimum cost if $d_2 > \Upsilon(d_1) + \eta$ and $d_1 < d_2 - \eta$.

Case B.IV. (24) holds for both $k = 1, 2$. In this case, TP2 cannot lead to the minimum cost, hence, the result is the same as the case $h > r$.

Then, for the case where $d_k = 2$ and $d_\ell < 2$ for $k, \ell = 1, 2, k \neq \ell$, the result in Case A still holds, since the analysis is independent of the magnitude of $\hat{q}$. Hence, for $k = 1$, for any $\eta > 0$, there exists an integer N_η such that for all $N > N_\eta$, $\Omega(RP)$ has the minimum cost if $d_1 < 2 - \eta$; either $\Omega(RP)$ or $\Omega(PO)$ has the minimum cost if $d_1 > 2 - \eta$. For $k = 2$, for any $\eta > 0$, there exists an integer N_η such that for all $N > N_\eta$,$\Omega(PR)$ has the minimum cost if $d_1 \leq 2 - \eta$; either $\Omega(PR)$ or $\Omega(OP)$ has the minimum cost if $d_1 \geq 2 - \eta$.

Combining the results in Case B, for any $\eta > 0$, there exists an integer N_η, such that for all $N > N_\eta$,(a) $\Omega(RR)$ is the LRE if $(d_1, d_2) \in B_1(\eta)$; (b) $\Omega(RP)$ is the LRE if $(d_1, d_2) \in B_2(\eta)$; (c) $\Omega(PR)$ is the LRE if $(d_1, d_2) \in B_3(\eta)$. By Definition 1, we have the statement in the theorem for $\hat{q} < \bar{q}$.

Furthermore, we can identify the LRE in the vanishing area $U(\eta)$. For any $\eta > 0$, there exists an integer N_η, such that the LRE set forms a subset of $\Omega(RP) \cup \Omega(RR)$ for $(d_1, d_2) \in U_a(\eta)$; the LRE set forms a subset of $\Omega(PR) \cup \Omega(RR)$ for $(d_1, d_2) \in U_b(\eta)$; the LRE set forms a subset of $\Omega(PR) \cup \Omega(RP)$ has the minimum cost for $(d_1, d_2) \in U_d(\eta)$; the LRE set forms a subset of $\Omega(PR) \cup \Omega(RP) \cup \Omega(PP) \cup \Omega(PO) \cup \Omega(OP)$ for $(d_1, d_2) \in U_c(\eta)$; and, the LRE set forms a subset of $\Omega(RR) \cup \Omega(RP) \cup \Omega(PR)$ for $(d_1, d_2) \in U_e(\eta)$. □

Corollary 3. *Let $h < r$. For any $\eta > 0$, there exists an integer $\bar{N}$, such that for all $N > \bar{N}$,*

(a) if $\hat{q} > 1/q^* - 2 + 2q^*$, for $(d_1, d_2) \in U(\eta)$, the LRE form a subset of

(a-i) $\Omega(RP) \cup \Omega(RR)$ if $(d_1, d_2) \in U_a(\eta)$;

(a-ii) $\Omega(PR) \cup \Omega(RR)$ if $(d_1, d_2) \in U_b(\eta)$;

(a-iii) $\Omega(PR) \cup \Omega(RP) \cup \Omega(OP) \cup \Omega(PO) \cup \Omega(PP)$ if $(d_1, d_2) \in U_c(\eta) \setminus \{(2, 2)\}$;

(a-iv) $\Omega(PR) \cup \Omega(RP)$ if $(d_1, d_2) \in U_d(\eta)$; and

(a-v) $\Omega(PR) \cup \Omega(RP) \cup \Omega(RR)$ if $(d_1, d_2) \in U_e(\eta)$.

(a-vi) Further, the LRE are the elements in $\Omega(PP)$ if $d_1 = d_2 = 2$.

(b) if $\hat{q} \leq 1/q^* - 2 + 2q^*$, for $(d_1, d_2) \in V(\eta)$, the LRE are the same as in Lemma 2.

Proof of Corollary 3. The results directly follow the proof of Theorem 2. □

Proof of Theorem 3. For any $\eta > 0$, denote

$$\begin{aligned}
D_1(\eta) &= \{(d_1, d_2) | d_1 > \Psi(d_2) + \eta, d_2 < d^* - \eta\} \\
D_2(\eta) &= \{(d_1, d_2) | d_2 > \Psi(d_1) + \eta, d_1 < d^* - \eta\} \\
D_3(\eta) &= \{(d_1, d_2) | d_1 > d^* + \eta, d_2 > d^* + \eta\} \\
G_1(\eta) &= \{(d_1, d_2) | d_1 > \Upsilon(d_2) + \eta, d_2 < \tilde{d} - \eta\} \\
G_2(\eta) &= \{(d_1, d_2) | d_2 > \Upsilon(d_1) + \eta, d_1 < \tilde{d} - \eta\} \\
G_3(\eta) &= \{(d_1, d_2) | d_1 > \tilde{d} + \eta, d_2 > \tilde{d} + \eta\}
\end{aligned}$$

Here we prove a stronger statement than the one in the theorem. That is, for $h \geq r$ or $\hat{q} \leq \bar{q}$, for any $\eta > 0$, there exists an integer N_η, such that for all $N \geq N_\eta$,

(a-i) any $(d_1, d_2) \in (A_1(\eta) \cup D_1(\eta) \cup D_2(\eta))$ corresponds to at least one NE; and

(a-ii) if $(d_1, d_2) \in D_3(\eta)$, there is no NE corresponding to (d_1, d_2).

For $h < r$ and $\hat{q} > \bar{q}$, for any $\eta > 0$, there exists an integer N_η, such that for all $N \geq N_\eta$,

(b-i) any $(d_1, d_2) \in (B_1(\eta) \cup G_1(\eta) \cup G_2(\eta))$ corresponds to at least one NE; and

(b-ii) if $(d_1, d_2) \in G_3(\eta)$, there is no NE corresponding to (d_1, d_2).

Consider first the case where $h \geq r$ or $\hat{q} \leq \bar{q}$. The LRE in this case are given in Theorem 1 and Lemma 2. $(d_1, d_2) \in D_1(\eta)$ leads to the selection of the element in $\Omega(RP)$ in the long run. Planner 2 has no incentive to change d_1 or d_2 by changing (c_2, p_2), because the individuals in location 2 are coordinating on the Pareto-efficient equilibrium. Planner 1 has no incentive to change only d_1, because changing d_1 can only move the LRE from $\Omega(RP)$ to $\Omega(RR)$. In either case, the individuals in location 1 would coordinate on the risk-dominant equilibrium. However, planner 1 has an incentive to increase d_2, because if d_2 is large enough, the LRE would become $\Omega(PR)$. Note that planner 1 cannot directly change d_2. The only possible way to change d_2 is to change p_1, because $d_2 = \min\{c_2, 2 - p_1\}$. Decreasing d_2 is always feasible, because planner 1 can increase p_1 and make $2 - p_1 = d_2$. However, increasing d_2 is *not* always feasible. $2 - p_1$ will increase by decreasing p_1, but, as long as $2 - p_1 > c_2$, $d_2 = c_2$, and decreasing p_1 cannot increase d_2 any more. Hence, let $c_2 = d_2$ and $2 - p_1 \geq d_2$. Then planner 1 has no incentive to change d_2. Therefore, any strategy profile $((c_1, p_1), (c_2, p_2))$ projected on $D_1(\eta)$ such that $d_1 = \min\{c_1, 2 - p_2\}$, $d_2 = c_2$ and $2 - p_1 \geq d_2$ is a Nash equilibrium.

$(d_1, d_2) \in D_2(\eta)$ leads to the selection of $\Omega(PR)$ in the long run. The argument is the same as above. Planner 1 has no incentive to change his strategy. By setting $d_1 = c_1 < 2-p_2$, planner 2 has no incentive to deviate either. Hence, for $(d_1, d_2) \in D_2(\eta)$, any strategy profile $((c_1, p_1), (c_2, p_2))$ projected on $D_2(\eta)$ such that $d_2 = \min\{c_2, 2-p_1\}$, $d_1 = c_1$ and $2-p_2 \geq d_1$ is a Nash equilibrium.

If $(d_1, d_2) \in A_1(\eta)$, the LRE are the elements in $\Omega(RR)$. Each planner k would have an incentive to increase d_ℓ ($\ell \neq k$) by decreasing p_k. However, this effort would be ineffective if $d_\ell = c_\ell < 2 - p_k$. Hence, any strategy profile $((c_1, p_1), (c_2, p_2))$ projected on $A_1(\eta)$ such that $d_k = c_k$ and $2 - p_\ell \geq d_k$ for both $k = 1, 2$, $\ell \neq k$, is a Nash equilibrium.

Lastly, we consider the area $D_3(\eta)$. If $(d_1, d_2) \in D_3(\eta) \cap A_2(\eta)$, the LRE is the element in $\Omega(RP)$. In this case, planner 1 will have an incentive to decrease c_1 so that $d_1 < d_2 - \eta$. It changes the LRE to $\Omega(PR)$ and increases the social welfare of location 1.

If $(d_1, d_2) \in D_3(\eta) \cap A_3(\eta)$, the LRE is the element in $\Omega(PR)$. In this case, planner 2 will have an incentive to decrease c_2 in such a way that $d_2 < d_1 - \eta$. This then leads to the selection of $\Omega(RP)$ in the long run, and increases the social welfare of location 2.

If $(d_1, d_2) \in D_3(\eta) \cap V_d(\eta)$, the LRE form a subset of $\Omega(RP) \cup \Omega(PR)$. We have shown that any strategy profile leading to the element in $\Omega(RP)$ or $\Omega(PR)$ is not a NE. Hence, any strategy profile leading to the selection of the elements in both $\Omega(RP)$ and $\Omega(PR)$ in the long run is not a NE either, because each social planner k will have an incentive to decrease c_k for $k = 1, 2$.

If $d_1 = d_2 = 2$, the elements in both $\Omega(PO)$ and $\Omega(OP)$ will be selected in the long run. Each of them will occur with probability 1/2. Hence, each social planner k will have an incentive to decrease c_k so that $d_k < d_\ell - \eta$ ($k \neq \ell$). Then, the players in location k will coordinate on the efficient equilibrium with probability one, and the social welfare of location k will increase.

If $(d_1, d_2) \in V_c(\eta) \setminus \{(2, 2)\}$, the LRE form a subset of $\Omega(OP) \cup \Omega(RP) \cup \Omega(PR) \cup \Omega(PO) \cup \Omega(PP)$. Note that there are no LRE which only consist of the elements in $\Omega(PP)$. If the elements in $\Omega(PP)$ are selected, the element in either $\Omega(RP)$ or $\Omega(PR)$ (or both) will be selected as well. Then, in any possible subset of the set above, with positive probability, at least one location will either have players coordinating on the less efficient equilibrium or have no players at all. Then, the social planner in this location k can always improve the social welfare by decreasing c_k so that $d_k < d_\ell - \eta$ ($k \neq \ell$). Therefore, no strategy profile projected on this area is a NE.

An analogous argument holds for the case with $h < r$ and $\hat{q} > \bar{q}$. The LRE for this case are given in Theorem 2 and Lemma 3. We only have to replace $D_1(\eta)$, $D_2(\eta)$, $D_3(\eta)$, $A_2(\eta)$, and $A_3(\eta)$ by $G_1(\eta)$, $G_2(\eta)$, $G_3(\eta)$, $B_2(\eta)$, and $B_3(\eta)$ respectively in the analysis above. Hence we obtain the result in the theorem. $\square$

Proof of Theorem 4. Again, we prove a stronger statement than the one in the theorem. That is, for any $\eta > 0$, there exists an integer N_η, such that for all $N \geq N_\eta$,

(a) $d_1 = d_2 = 1$ corresponds to at least one NE;

(b) there is no NE corresponding to $(d_1, d_2) \in [1, 2]^2 \setminus (\{(1, 1)\} \cup E(\eta))$, where $E(\eta) = V_e(\eta)$ if $h \geq r$ or $\hat{q} \leq \bar{q}$, and $E(\eta) = U_e(\eta)$ if $h < r$ and $\hat{q} > \bar{q}$.

Consider a strategy profile $((c_1, p_1), (c_2, p_2))$ such that $c_1 = c_2 = 1$ and $p_1 = p_2 = 1$. Planner $k = 1, 2$ has no incentive to deviate from his strategy. Changing c_k has no effect, because all the individuals in location $\ell \neq k$ are immobile, hence cannot move to location k. Changing p_1 has no effect either, because the maximum capacity of location ℓ is N, hence, the mobile players in location k cannot move to location ℓ. Hence, this strategy profile is a Nash equilibrium, which corresponds to $d_1 = d_2 = 1$.

For $h \geq r$ or $\hat{q} \leq \bar{q}$, consider any $(d_1, d_2) \in [1, 2]^2 \setminus (\{(1, 1)\} \cup V_e(\eta))$. We first claim that any strategy profile projected on $A_1(\eta) \setminus \{(1, 1)\}$ is not a NE. In this area, the LRE are the elements in $\Omega(RR)$. The social planner of location k will always have an incentive to decrease d_ℓ ($\ell \neq k$) by increasing p_k. The reason is that the population in location k fluctuates between m_k and M_k. Decreasing d_ℓ will increase the lower bound of the population in location k, hence improving the social welfare.

For $(d_1, d_2) \in A_2(\eta)$, the LRE is the element in $\Omega(RP)$, and the population in location 1 is $2N - \lfloor d_2 N \rfloor$. The social planner in location 1 will have an incentive to decrease d_1 by setting a lower c_1, so that the LRE form a subset of $\Omega(RR) \cup \Omega(PR)$. Denote by d'_k the parameter of effective capacity of location $k = 1, 2$ after a deviation. If such a deviation leads the elements in $\Omega(RR)$ to be selected, the population in location 1 will fluctuate between $2N - \lfloor d_2 N \rfloor$ and $\lfloor d'_1 N \rfloor$, hence the social welfare will increase. If the deviation leads $\Omega(PR)$ to be selected, the players in location 1 will coordinate on P, and the population will increase to $\lfloor d'_1 N \rfloor$, which improves the social welfare of location 1. Based on the results above, any deviation that results in the selection of the elements in $\Omega(RR)$ and $\Omega(PR)$ also increases the social welfare of location 1. Hence, no strategy profile projected on $A_2(\eta)$ is a NE. Symmetrically, no strategy profile projected on $A_3(\eta)$ is a NE. The argument is analogous to the case above. Here, the social planner of location 2 will always have an incentive to decrease d_2. Such a deviation can at least increase the population in location 2, hence improving social welfare.

If $(d_1, d_2) \in V_a(\eta)$, $\Psi(d_2) - \eta \leq d_1 \leq \Psi(d_2) + \eta$. The LRE form a subset of $\Omega(RR) \cup \Omega(RP)$. We have argued that if the LRE are the elements in $\Omega(RR)$ or $\Omega(RP)$, the social planner of location 1 will always have an incentive to deviate. If the LRE are the elements in $\Omega(RR)$ and $\Omega(RP)$, with positive probability the population in location 1 will fluctuate between $2N - \lfloor d_2 N \rfloor$ and $\lfloor d_1 N \rfloor$, and with the remaining probability the population in location 1 is $2N - \lfloor d_2 N \rfloor$. In this case, the social planner of location 1 will always have an incentive to decrease c_1 so that d'_1 is smaller than but arbitrarily close to $\Psi(d_2) - \eta$. Then, the LRE are the elements in $\Omega(RR)$ and the population in location 1 will fluctuate between $2N - \lfloor d_2 N \rfloor$ and $\lfloor (d_1 - \eta) N \rfloor$ with probability one. For η small enough, this will increase the social welfare of location 1. Symmetrically, for $(d_1, d_2) \in V_b(\eta)$, the same argument holds for the social planners of location 2. Hence, he will have an incentive to decrease d_2.

If $(d_1, d_2) \in V_d(\eta)$, the LRE form a subset of $\Omega(RP) \cup \Omega(PR)$. We have shown above that the strategy profiles which lead to the selection of the element in either $\Omega(RP)$ or $\Omega(PR)$ are not NE. If the LRE are $\Omega(RP)$ and $\Omega(PR)$, the expected population in each location k should fall in the interval $]2N - \lfloor d_\ell N \rfloor, \lfloor d_k N \rfloor[$ where $d_\ell - \eta \leq d_k \leq d_\ell + \eta$, and the individuals in location k will either coordinate on R or P. The social planner of location k will have an incentive to decrease c_k, so that d'_k is smaller than but arbitrarily close to $d_\ell - \eta$. Then, the players in location k will coordinate on P with probability one, and the population will be arbitrarily close to $\lfloor (d_k - \eta) N \rfloor$. For η small enough, this deviation will increase the social welfare of location k.

If $d_1 = d_2 = 2$, the elements in both $\Omega(PO)$ and $\Omega(OP)$ will be selected in the long run. Each of them will occur with probability $1/2$. Hence, the average expected payoff of location k is lower than the payoff of the Pareto-efficient equilibrium, and the expected population will fall in the interval $]0, 2N[$. The social planner of location $k = 1, 2$ will have an incentive to decrease c_k so that d'_k is smaller than but arbitrarily close to $2 - \eta$. Then, the players in location k will coordinate on P with probability one, and the population of location k will be infinitely close to $\lfloor (2 - \eta) N \rfloor$. For η small enough, this will increase the social welfare of location k.

If $(d_1, d_2) \in (V_c(\eta) \setminus \{(2, 2)\})$, $d_k \in (2 - \eta, 2)$ for both $k = 1, 2$, and the LRE form a subset of $\Omega(OP) \cup \Omega(RP) \cup \Omega(PR) \cup \Omega(PO) \cup \Omega(PP)$. The same argument applies to show that the strategy profiles leading to the selection of the element in an singleton absorbing set ($\Omega(RP), \Omega(PR), \Omega(OP)$ or $\Omega(PO)$) are not NE. We have pointed out in the proof of Theorem 3 that if the elements in $\Omega(PP)$ are selected, the element(s) in either $\Omega(RP)$ or $\Omega(PR)$ (or both) must be selected as well in the long run. Hence, in any possible subset of the set above, with positive probability, at least one location k will either have players coordinating on R or have no players at all. In such a case, the average expected payoff for the players in location k will be less than that of the Pareto-efficient equilibrium, and the expected population will be in the interval $]0, \lfloor d_k N \rfloor[$. Hence, the social planner of location k will have an incentive to decrease c_k so that $d'_k = d_k - \eta$. Then, the players in location k will coordinate on P with probability one, and the population will be arbitrarily close to $\lfloor (d_k - \eta) N \rfloor$. For η small enough, this deviation will increase the social welfare of location k. The analysis is analogous for the case with $h < r$ and $\hat{q} > \bar{q}$. Hence we have the result in the statement. $\square$

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