Robust Resampling Methods for Time Series*

Lorenzo Camponovo

University of Lugano

Olivier Scaillet

University of Geneva and Swiss Finance Institute

Fabio Trojani

University of Lugano and Swiss Finance Institute

First Version: November 2007; This Version: August 31, 2011

*The authors acknowledge the financial support of the Swiss National Science Foundation (NCCR FINRISK and grants 101312-103781/1, 100012-105745/1, and PDFM1-114533). We thank participants at the International Conference on Computational Management Science 2009 in Geneva, the International Conference on Robust Statistics 2009 in Parma, the International Conference on Computational and Financial Econometrics 2009 in Limassol, the Prospectus Workshop in Econometrics 2010 at Yale University, the seminar at Gent University March 2010, the Conference on Resampling Methods and High Dimensional Data 2010 at Texas A&M University, and ECARES seminar at ULB April 2010 in Bruxelles for helpful comments. Part of this research was done when the first author was visiting Yale University as a post doctoral researcher. Correspondence address: Fabio Trojani, Faculty of Economics, University of Lugano, Via Buffi 13, CH-6900 Lugano, e-mail: Fabio.Trojani@usi.ch.

Abstract

We study the robustness of block resampling procedures for time series. We derive a set of formulas to characterize their quantile breakdown point. For the moving block bootstrap and the subsampling, we find a very low quantile breakdown point. A similar robustness problem arises in relation to data-driven methods for selecting the block size in applications. This can render inference based on standard resampling methods virtually useless already in simple estimation and testing settings. To solve this problem, we introduce a robust fast resampling scheme that is applicable to a wide class of time series models. Monte Carlo simulations and sensitivity analysis for the simple AR(1), both stationary and near-to-unit root settings, confirm the dramatic fragility of classical resampling procedures in presence of contaminations by outliers. They also show the better accuracy and efficiency of the robust resampling approach under different types of data constellations. A real data application to testing for stock return predictability shows that our robust approach can detect predictability structures more consistently than classical methods.

Keywords: Bootstrap, subsampling, breakdown point, robustness, time series.

JEL: C12, C13, C15.

MSC 2000: Primary 62F40; Secondary 62F35.

1 Introduction

Resampling methods, including the bootstrap (see, e.g., Hall, 1992, Efron and Tibshirani, 1993, and Hall and Horowitz, 1996) and the subsampling (see, e.g., Politis and Romano, 1992, 1994a, Politis, and Romano and Wolf, 1999), are useful tools in modern statistics and econometrics. The simpler consistency conditions and the wider applicability in some cases (see, e.g., Andrews, 2000, and Bickel, Gotze and van Zwet, 1997) make the subsampling a useful and valid alternative to the bootstrap in a number of statistical models. Bootstrap and subsampling procedures for time series typically rely on different block resampling schemes, in which selected sub-blocks of the data, having size strictly less than the sample size, are randomly resampled. This feature is necessary in order to derive consistent resampling schemes under different assumptions on the asymptotically vanishing time series dependence between observations. See, among others, Hall (1985), Carlstein (1986), Künsch (1989), Politis, Romano and Wolf (1999), Bühlmann (2002), and Lahiri (2003).

The low robustness of classical bootstrap and subsampling methods is a known feature in the iid setting; see, e.g., Singh (1998), Salibian-Barrera and Zamar (2002), Salibian-Barrera, Van Aelst and Willems (2006, 2007), and Camponovo, Scaillet and Trojani (2010). These papers study global robustness features and highlight a typically very low breakdown point of classical bootstrap and subsampling quantiles. Moreover, they show that this robustness problem cannot be exhausted simply by applying classical bootstrap and subsampling methods to high breakdown point statistics. Essentially, the breakdown point quantifies the smallest fraction of outliers in the data which makes a statistic meaningless. Therefore, iid resampling methods produce estimated quantiles that are heavily dependent on a few possible outliers in the original data. Intuitively, this lack of robustness is related to the (typically high) probability of resampling a large number of outliers in a random sample using an iid bootstrap or subsampling scheme. To overcome this problem, robust bootstrap and subsampling approaches with desirable quantile breakdown point properties have been developed in the iid context by Salibian-Barrera and Zamar (2002), Salibian-Barrera, Van Aelst and Willems (2006, 2007), and Camponovo, Scaillet and Trojani (2010), among others.

In this paper, we study the robustness of block resampling methods for time series and develop fast robust resampling approaches that are applicable to a variety of time series models. We first characterize the breakdown properties of block resampling procedures for time series by deriving lower and upper bounds for their quantile breakdown point; these results cover both overlapping and nonoverlapping bootstrap and subsampling procedures. Concrete computations show that block resampling methods for time series suffer of an even larger robustness problem than in the iid context. This problem cannot be mitigated simply by applying standard block resampling methods to a more robust statistic (for simple examples see Singh (1998) in applying the bootstrap, and Camponovo, Scaillet and Trojani (2010) in applying the subsampling): resampling a robust statistic does not yield robust resampling! This indicates the high need for a more robust resampling scheme applicable in the time series context.

We develop our robust resampling approach for time series following the fast resampling idea put forward, among others, in Shao and Tu (1995), Davidson and McKinnon (1999), Hu and Kalbfleisch (2000), Andrews (2002), Salibian-Barrera and Zamar (2002), Goncalves and White (2004), Hong and Scaillet (2006), Salibian-Barrera, Van Aelst and Willems (2006, 2007), and Camponovo, Scaillet and Trojani (2010). Our resampling method is applicable to a wide class of resampling procedures, including both the block bootstrap and the subsampling; it provides robust estimation and inference results under weak conditions. Moreover, it inherits the low computational cost of fast resampling approaches. This makes it applicable to nonlinear models when classical resampling methods might become computationally too expensive, when applied to robust statistics or in combination with computationally intensive data-driven procedures for the selection of the optimal block size; see, among others, Sakata and White (1998), Ronchetti and Trojani (2001), Mancini, Ronchetti and Trojani (2005), Ortelli and Trojani (2005), Muler and Yohai (2008), and La Vecchia and Trojani (2010) for recent examples of robust estimators for nonlinear time series models. By means of explicit breakdown point computations, we also find that the better breakdown properties of our fast robust resampling scheme are inherited by data-driven choices of the block size based on either (i) the minimum confidence index volatility (MCIV) and the calibration method (CM), proposed in Romano and Wolf (2001) for the subsampling, or (ii) the data-driven method in Hall, Horowitz and Jing (1995) (HHJ) for the moving block bootstrap.

We investigate by Monte Carlo simulation the performance of our robust resampling approach in the benchmark context of the estimation of the autoregressive parameter in an AR(1) model, both in stationary and near-to-unit root settings. Overall, our Monte Carlo experiments highlight a dramatic fragility of classical resampling methods in presence of contaminations by outliers, and a more reliable and efficient inference produced by our robust resampling method under different types of data constellations. Finally, in an application to real data, we find that our robust resampling approach detects predictability structures in stock returns more consistently than standard methods.

The paper is organized as follows. Section 2 oulines the main setting and introduces the quantile breakdown point formulas of different block resampling procedures in time series. In Section 3, we develop our robust approach and derive the relevant formula for the associated quantile breakdown point. We show that, under weak conditions, the resulting quantile breakdown point is maximal. In Section 4, we study the robustness properties of data-driven block size selection procedures based on the MCIV, the CM, and the HHJ method. Monte Carlo experiments, sensitivity analysis and the empirical application to stock returns predictability are presented in Section 5. Section 6 concludes.

2 Resampling Distribution Quantile Breakdown Point

We start our analysis by characterizing the robustness of resampling procedures for time series and by deriving formulas for their quantile breakdown point.

2.1 Definition

Let $X_{(n)} = (X_1, \ldots, X_n)$ be an observation sample from a sequence of random vectors with $X_i \in \mathbb{R}^{d_x}$, $d_x \geq 1$, and consider a real valued statistic $T_n := T(X_{(n)})$.

In the time series setting, block bootstrap procedures split the original sample in overlapping or nonoverlapping blocks of size $m < n$. Then, new random samples of size n are constructed assuming an approximate independence between blocks. Finally, the statistic T is applied to the so generated random samples; see, e.g., Hall (1985), Carlstein (1986), Künsch (1989), and Andrews (2004). The more recent subsampling method (see, e.g., Politis, Romano and Wolf, 1999), instead directly applies statistic T to overlapping or nonoverlapping blocks of size m strictly less than n.

Let $X_{(n,m)}^{K*} = (X_1^*, \ldots, X_n^*)$ denote a nonoverlapping ($K = NB$) or an overlapping ($K = OB$) block bootstrap sample, constructed using nonoverlapping or overlapping blocks of size m, respectively. Similarly, let $X_{(n,m)}^{K*} = (X_1^*, \ldots, X_m^*)$ denote a nonoverlapping ($K = NS$) or an overlapping ($K = OS$) subsample, respectively. We introduce the nonoverlapping and overlapping block bootstrap and subsampling statistics $T_{n,m}^{K*} := T(X_{(n,m)}^{K*})$, where $K = NB, OB, NS, OS$, respectively. Then, for $t \in (0, 1)$, the quantile $Q_{t,n,m}^{K*}$ of $T_{n,m}^{K*}$ is defined by

$$Q_{t,n,m}^{K*} = \inf\{x | P^*(T_{n,m}^{K*} \leq x) \geq t\}, \tag{1}$$

where P^* is the corresponding bootstrap or subsampling distribution and, by definition, $\inf(\emptyset) = \infty$.

We characterize the robustness of the quantile defined in (1) via its breakdown point, i.e., the smallest fraction of outliers in the original sample such that $Q_{t,n,m}^{K*}$ degenerates, making inference based on (1) meaningless. Different than in the iid case, in time series we can consider different possible models of contamination by outliers, like for instance additive outliers, replacement outliers, and innovation outliers; see, e.g., Martin and Yohai (1986). Because of this additional complexity, we first introduce a notation that can better capture the effect of such contaminations, following Genton and Lucas (2003). Denote by $\mathcal{Z}_{n,p}^{\zeta}$ the set of all n-components outlier samples, where p is the number of the d_x-dimensional outliers, and index $\zeta \in \bar{\mathbb{R}}^{d_x}$ indicates their size. When $p > 1$, we do not necessarily assume outliers $\zeta_1, \ldots, \zeta_p$ to be all equal to ζ, but we rather assume existence of constants $c_1, \ldots, c_p$, such that $\zeta_i = c_i \zeta$.

Let $0 \leq b \leq 0.5$ be the upper breakdown point of statistic T_n, i.e., nb is the smallest number of outliers such that $T(X_{(n)} + Z_{n,nb}^{\zeta}) = +\infty$ for some $Z_{n,nb}^{\zeta} \in \mathcal{Z}_{n,nb}^{\zeta}$. The breakdown point b is an intrinsic characteristic of a statistic. It is explicitly known in some cases and it can be gauged most of the time, for instance by means of simulations and sensitivity analysis. In this section, we focus for brevity on one-dimensional real valued statistics. As discussed for instance by Singh (1998) in the iid context, our quantile breakdown point results for time series can be naturally extended to consider multivariate and scale statistics. Formally, the quantile breakdown point of $Q^{K*}_{t,n,m}$ is defined as follows:

Definition 1 *The upper breakdown point of the t-quantile* $Q^{K*}_{t,n,m} := Q^{K*}_{t,n,m}(X_{(n)})$ *is given by*

$$b^K_{t,n,m} := \frac{1}{n}\cdot\left[\inf_{\{1\le p\le\lceil n/2\rceil\}}\left\{p\middle|\text{there exists } Z^{\zeta}_{n,p}\in\mathcal{Z}^{\zeta}_{n,p} \text{ such that } Q^{K*}_{t,n,m}(X_{(n)}+Z^{\zeta}_{n,p})=+\infty\right\}\right], \quad (2)$$

where $\lceil x\rceil = \inf\{n\in\mathbb{N}|x\le n\}$.

2.2 Quantile Breakdown Point

We derive lower and upper bounds for the quantile breakdown point of the overlapping subsampling and both nonoverlapping and overlapping moving block bootstrap procedures. Similar results can be obtained for the nonoverlapping subsampling. Since that case is of little practical interest, because unless the sample size is very large the number of blocks is too small to make reliable inference, we do not report results for such a case. Results for the overlapping moving block bootstrap can be modified to cover asymptotically equivalent variations, such as the stationary bootstrap of Politis and Romano (1994b).

2.2.1 Subsampling

For simplicity, let $n/m = r\in\mathbb{N}$. The overlapping subsampling splits the original sample $X_{(n)} = (X_1,\ldots,X_n)$ into $n-m+1$ overlapping blocks of size m, $(X_i,\ldots,X_{i+m-1})$, $i=1,\ldots,n-m+1$. Finally, it applies statistic T to these blocks.

Theorem 2 *Let b be the breakdown point of T_n and $t\in(0,1)$. The quantile breakdown point* $b^{OS}_{t,n,m}$ *of overlapping subsampling procedures satisfies the following property:*

$$\frac{\lceil mb\rceil}{n}\le b^{OS}_{t,n,m}\le \inf_{\{p\in\mathbb{N},p\le r-1\}}\left\{p\cdot\frac{\lceil mb\rceil}{n}\middle|p>\frac{(1-t)(n-m+1)+\lceil mb\rceil-1}{m}\right\}. \quad (3)$$

The term $\frac{(1-t)(n-m+1)}{m}$ represents the number of degenerated subsampling statistics necessary in order to cause the breakdown of $Q^{OS*}_{t,n,m}$, while $\frac{\lceil mb \rceil}{n}$ is the fraction of outliers which is sufficient to cause the breakdown of statistic T in a block of size m. In time series, the number of possible subsampling blocks of size m is typically lower than the number of iid subsamples of size m. Therefore, the breakdown of a statistic in one random block tends to have a larger impact on the subsampling quantile than in the iid case. Intuitively, this feature implies a lower breakdown point of subsampling quantiles in time series than in iid settings. Table 1 confirms this basic intuition. Using Theorem 2, we compute lower and upper bounds for the breakdown point of the overlapping subsampling quantile for a sample size $n = 120$, for $b = 0.5$, and for block sizes $m = 5, 10, 20$. We see that even for a maximal breakdown point statistic ($b = 0.5$), the overlapping subsampling implies a very low quantile breakdown point, which is increasing in the block size, but very far from the maximal value $b = 0.5$. Moreover, this breakdown point is clearly lower than in the iid case; see Camponovo, Scaillet and Trojani (2010). For instance, for $m = 10$, the 0.95-quantile breakdown point of the overlapping subsampling is 0.0417, which is less than a quarter of the breakdown point of 0.23 for the same block size in the iid setting.

2.2.2 Moving Block Bootstrap

Let $X^N_{(m),i} = (X_{(i-1)\cdot m+1}, \ldots, X_{i\cdot m})$, $i = 1, \ldots, r$, be the r nonoverlapping blocks of size m. The nonoverlapping moving block bootstrap selects randomly with replacement r nonoverlapping blocks $X^{N*}_{(m),i}$, $i = 1, \ldots, r$. Then, it applies statistic T to the n-sample $X^{NB*}_{(n,m)} = (X^{N*}_{(m),1}, \ldots, X^{N*}_{(m),r})$. Similarly, let $X^O_{(m),i} = (X_i, \ldots, X_{i+m-1})$, $i = 1, \ldots, n-m+1$, be the $n-m+1$ overlapping blocks. The overlapping moving block bootstrap selects randomly with replacement r overlapping blocks $X^{O*}_{(m),i}$, $i = 1, \ldots, r$. Then, it applies statistic T to the n-sample $X^{OB*}_{(n,m)} = (X^{O*}_{(m),1}, \ldots, X^{O*}_{(m),r})$.

Theorem 3 *Let b be the breakdown point of T_n, $t \in (0,1)$, and $p_1, p_2 \in \mathbb{N}$, with $p_1 \leq m, p_2 \leq r-1$. The quantile breakdown points $b^{NB}_{t,n,m}$ and $b^{OB}_{t,n,m}$ of the nonoverlapping and overlapping moving block bootstrap, respectively, satisfy the following properties:*

(i) $\frac{\lceil mb \rceil}{n} \leq b^{NB}_{t,n,m} \leq \frac{1}{n} \cdot \left[\inf_{\{p_1,p_2\}} \left\{ p = p_1 \cdot p_2 \middle| P\left(BIN\left(r, \frac{p_2}{r}\right) \geq \lceil \frac{nb}{p_1} \rceil\right) > 1-t \right\} \right]$,

(ii) $\frac{\lceil mb \rceil}{n} \leq b^{OB}_{t,n,m} \leq \frac{1}{n} \cdot \left[\inf_{\{p_1,p_2\}} \left\{ p = p_1 \cdot p_2 \middle| P\left(BIN\left(r, \frac{mp_2 - p_1 + 1}{n-m+1} \right) \geq \lceil \frac{nb}{p_1} \rceil \right) > 1 - t \right\} \right]$.

The right part of (i) and (ii) are similar for large $n >> m$. Indeed, (ii) implies $\frac{mp_2 - p_1 + 1}{n-m+1} \approx \frac{mp_2}{n} = \frac{p_2}{r}$, which is the right part of (i). Further the breakdown point formula for the iid bootstrap in Singh (1998) emerges as a special case of the formulas in Theorem 3, for $m = 1$. This is intuitive: a nonoverlapping moving block bootstrap with block size m is essentially an iid bootstrap based on a sample of size r, in which each block of size m corresponds to a single random realization in the iid bootstrap. As for the subsampling, the reduction in the number of possible blocks when $m \neq 1$ increases the potential impact of a contamination and implies a lower quantile breakdown point. In Table 1, we compute lower and upper bounds for the breakdown point of the nonoverlapping and overlapping moving block bootstrap quantile for $n = 120$, $b = 0.5$, and block sizes $m = 5, 10, 20$. Again, they are far from the maximal value $b = 0.5$, and lower than in the iid case. For instance, for $m = 10$, the 0.99-quantile breakdown point is less than 0.25, which is smaller than the breakdown point of 0.392 in the iid setting.

3 Robust Resampling Procedures

The results in the last section show that, even using statistics with maximal breakdown point, classical block resampling procedures can imply a low quantile breakdown point. To overcome this problem, it is necessary to introduce a different and more robust resampling approach. We develop such robust resampling methods for M-estimators, starting from the fast resampling approach studied, among others, in Shao and Tu (1995), Davidson and McKinnon (1999), Hu and Kalbfleisch (2000), Andrews (2002), Salibian-Barrera and Zamar (2002), Goncalves and White (2004), Hong and Scaillet (2006), Salibian-Barrera, Van Aelst and Willems (2006, 2007), and Camponovo, Scaillet and Trojani (2010).

3.1 Definition

Given the original sample $X_{(n)} = (X_1, \dots, X_n)$, we consider the class of robust M-estimators $\hat{\theta}_n$ for parameter $\theta \in \mathbb{R}^d$, defined as the solution of the equations:

$$\psi_n(X_{(n)}, \hat{\theta}_n) := \frac{1}{n} \sum_{i=1}^{n} g(X_i, \hat{\theta}_n) = 0, \tag{4}$$

where $\psi_n(X_{(n)}, \cdot) : \mathbb{R}^d \to \mathbb{R}^d$ depends on parameter θ and for a bounded estimating function g. Boundedness of estimating function g is a characterizing feature of robust M-estimators, see e.g., Hampel, Ronchetti, Rousseeuw and Stahel (1986). Standard block resampling approaches need to solve equation $\psi_k(X^{K*}_{(n,m)}, \hat{\theta}^{K*}_{n,m}) = 0$, for each bootstrap ($k = n$, $K = NB, OB$) or subsampling ($k = m < n$, $K = OS$) random sample $X^{K*}_{(n,m)}$, which is computationally demanding. Instead, we consider the following Taylor expansion of (4) around the true parameter θ_0:

$$\hat{\theta}_n - \theta_0 = -[\nabla_\theta \psi_n(X_{(n)}, \theta_0)]^{-1} \psi_n(X_{(n)}, \theta_0) + o_p(1), \tag{5}$$

where $\nabla_\theta \psi_n(X_{(n)}, \theta_0)$ denotes the derivative of function ψ_n with respect to parameter θ. Based on this expansion, we use $-[\nabla_\theta \psi_n(X_{(n)}, \hat{\theta}_n)]^{-1} \psi_k(X^{K*}_{(n,m)}, \hat{\theta}_n)$ as an approximation of $\hat{\theta}^{K*}_{n,m} - \hat{\theta}_n$ in the definition of the resampling scheme estimating the sampling distribution of $\hat{\theta}_n - \theta_0$. This avoids computing $\hat{\theta}^{K*}_{n,m}$ and $[\nabla_\theta \psi_k(X^{K*}_{(n,m)}, \hat{\theta}_n)]^{-1}$ on resampled data.

Given a normalization constant τ_n, a robust fast resampling distribution for $\tau_n(\hat{\theta}_n - \theta_0)$ is defined by

$$L^{RF,K*}_{n,m}(x) = \frac{1}{N} \sum_{s=1}^{N} \mathbb{I}\Bigg(\tau_k \Bigg(-[\nabla_\theta \psi_n(X_{(n)}, \hat{\theta}_n)]^{-1} \psi_k(X^{K*}_{(n,m),s}, \hat{\theta}_n) \Bigg) \leq x \Bigg), \tag{6}$$

where $\mathbb{I}(\cdot)$ is the indicator function and s indexes the N possible random samples generated by subsampling and bootstrap procedures, respectively. The main assumptions under which the fast resampling distribution (6) consistently estimates the unknown sampling distribution of $\tau_n(\hat{\theta}_n - \theta_0)$ in a time series context are given, e.g., in Hong and Scaillet (2006) for the subsampling (Assumption 1) and in Goncalves and White (2004) for the bootstrap (Assumption

A and Assumptions 2.1 and 2.2).

Before analyzing the robustness properties of the robust fast resampling distribution (6), we provide a final remark on the rate of convergence. We denote by E^* the expectation with respect to the probability measure induced by the resampling method. As pointed out for instance in Hall et al. (1995), the overlapping scheme of classical resampling methods generally implies $E^*[\hat{\theta}^{K*}_{n,m}] \neq E^*[\hat{\theta}_n]$. Because of this distortion, the rate of convergence of the resampling distribution decreases. To overcome the problem, a simple solution consists in considering the recentered statistic $\hat{\theta}^{K*}_{n,m} - E^*[\hat{\theta}^{K*}_{n,m}]$, instead of $\hat{\theta}^{K*}_{n,m} - \hat{\theta}_n$. Note that the recentering approach can be applied with our robust fast procedure as well. Indeed, as shown by Andrews (2002) in relation to a fast bootstrap method, the robust fast recentered distribution

$$L^{RF,K*}_{n,m}(x) = \frac{1}{N}\sum_{s=1}^{N} \mathbb{I}\Bigg(\tau_k\Bigg(- [\nabla_\theta \psi_n(X_{(n)}, \hat{\theta}_n)]^{-1}(\psi_k(X^{K*}_{(n,m),s}, \hat{\theta}_n) - E^*[\psi_k(X^{K*}_{(n,m),s}, \hat{\theta}_n)])\Bigg) \leq x\Bigg), \tag{7}$$

implies by construction the nondistortion condition

$$E^*\Bigg[[\nabla_\theta \psi_n(X_{(n)}, \hat{\theta}_n)]^{-1}(\psi_k(X^{K*}_{(n,m)}, \hat{\theta}_n) - E^*[\psi_k(X^{K*}_{(n,m)}, \hat{\theta}_n)])\Bigg] = 0. \tag{8}$$

Using our robust approach, the computation of $E^*[\psi_k(X^{K*}_{(n,m)}, \hat{\theta}_n)]$ only marginally increases the computational costs, which are instead very large for the computation of $E^*[\hat{\theta}^{K*}_{n,m}]$ based on the classical resampling methods. Unreported Monte Carlo simulations show a better accuracy of the recentered procedures for both standard and robust approaches. In particular, the improvement produced by the recentering is more evident for nonsymmetric confidence intervals, while it is less pronounced for symmetric two-sided confidence intervals.

3.2 Robust Resampling Methods and Quantile Breakdown Point

In the computation of (6) we only need consistent point estimates for parameter vector θ_0 and matrix $-[\nabla_\theta \psi_n(X_{(n)}, \theta_0)]^{-1}$, based on the whole sample $X_{(n)}$. These estimates are given by $\hat{\theta}_n$ and $-[\nabla_\theta \psi_n(X_{(n)}, \hat{\theta}_n)]^{-1}$, respectively. Thus, a computationally very fast procedure is obtained. This feature is not shared by standard resampling schemes, which can easily become unfeasible when applied to robust statistics.

A closer look at $-[\nabla_\theta \psi_n(X_{(n)},\hat{\theta}_n)]^{-1}\ \psi_k(X^{K*}_{(n,m),s},\hat{\theta}_n)$ reveals that this quantity can degenerate to infinity when either (i) matrix $\nabla_\theta \psi_n(X_{(n)},\hat{\theta}_n)$ is singular or (ii) estimating function g is not bounded. Since we are making use of a robust (bounded) estimating function g, situation (ii) cannot arise. From these arguments, we obtain the following corollary.

Corollary 4 *Let b be the breakdown point of the robust M-estimator $\hat{\theta}_n$ defined by (4). The t-quantile breakdown point of resampling distribution (6) is given by $b^K_{t,n,m}=\min(b,b_{\nabla\psi})$, where*

$$b_{\nabla\psi}=\frac{1}{n}\cdot\inf_{1\le p\le\lceil n/2\rceil}\left\{p\middle|\text{there exists } Z^{\zeta}_{n,p}\in\mathcal{Z}^{\zeta}_{n,p}\text{ such that } det(\nabla_\theta\psi_n(X_{(n)}+Z^{\zeta}_{n,p},\hat{\theta}_n))=0\right\}. \quad (9)$$

The quantile breakdown point of our robust fast resampling distribution is the minimum of the breakdown point of M-estimator $\hat{\theta}_n$ and matrix $\nabla_\theta\psi_n(X_{(n)},\hat{\theta}_n)$. In particular, if $b_{\nabla\psi}\ge b$, the quantile breakdown point of our robust resampling distribution (6) is maximal, independent of confidence level t.

Unreported Monte Carlo simulations show that the application of our fast approach to a nonrobust estimating function only marginally provides some robustness improvements, and does not solve the robustness issue. It turns out that in order to ensure robustness both the fast approach and a robust bounded estimating function are necessary.

4 Breakdown Point and Data Driven Choice of the Block Size

A main issue in the application of block resampling procedures is the choice of the block size m, since accuracy of the resampling distribution depends strongly on this parameter; see, e.g., Lahiri, (2001). In this section, we study the robustness of data driven block size selection approaches for subsampling and bootstrap procedures. We first consider MCIV method and the CM for the subsampling. In a second step, we analyze the HHJ method for the bootstrap.

Given a sample of size n, we denote by $\mathcal{M}=\{m_{\min}\ldots,m_{\max}\}$ the set of admissible block sizes. The MCIV, the CM and HHJ select the optimal block size $m_u\in\mathcal{M}$, with

$u = MCIV, CM, HHJ$, respectively, as solution of a problem of the form

$$m_u = \arg\inf_{m\in\mathcal{M}}\{F_{u,1}(X_{(n)}; m) : F_{u,2}(X_{(n)}; m) \in I_u\}\ , \tag{10}$$

where by definition $\arg\inf(\emptyset) := \infty$, $F_{u,1}$, $F_{u,2}$ are two scalar functions of the original sample $X_{(n)}$ and block size m, and I_u is a subset of $\mathbb{R}$; see Equations (13), (15), and (18) below for the explicit definitions of $F_{u,1}$, $F_{u,2}$ in the setting $u = MCIV, CM, HHJ$, respectively.

For these methods, we characterize the smallest fraction of outliers in the original sample such that the data driven choice of the block size fails and diverges to infinity. More precisely, we compute the breakdown point of $m_u := m_u(X_{(n)})$, with $u = MCIV, CM, HHJ$, defined as

$$b_t^u = \frac{1}{n} \cdot \inf_{1\le p\le\lceil n/2\rceil} \left\{p \middle| \text{there exists } Z_{n,p}^\zeta \in \mathcal{Z}_{n,p}^\zeta \text{ such that } \ m_u(X_{(n)} + Z_{n,p}^\zeta) = \infty\right\}. \tag{11}$$

4.1 Subsampling

Denote by $b_t^{OS,J}$, $J = MCIV, CM$, the breakdown point of the overlapping subsampling based on MCIV and CM methods, respectively.

4.1.1 Minimum Confidence Index Volatility

A consistent method for a data driven choice of the block size m is based on the minimization of the confidence interval volatility index across the admissible values of m. For brevity, we present the method for one–sided confidence intervals. Modifications for two–sided confidence intervals are obvious.

Definition 5 *Let $m_{\min} < m_{\max}$ and $k \in \mathbb{N}$ be fixed. For $m \in \{m_{\min} - k, .., m_{\max} + k\}$, denote by $Q_t^*(m)$ the $t-$subsampling quantile for the block size m. Further, let $\overline{Q}_t^{*k}(m)$ be the average quantile $\overline{Q}_t^{*k}(m) := \frac{1}{2k+1}\sum_{i=-k}^{i=k} Q_t^*(m+i)$. The confidence interval volatility (CIV) index is defined for $m \in \{m_{\min}, ..., m_{\max}\}$ by*

$$CIV(m) := \frac{1}{2k+1}\sum_{i=-k}^{i=k}\left(Q_t^*(m+i) - \overline{Q}_t^{*k}(m)\right)^2 . \tag{12}$$

Let $\mathcal{M} := \{m_{\min}, \ldots, m_{\max}\}$. The data driven block size that minimizes the confidence interval volatility index is

$$m_{MCIV} = \arg\inf_{m\in\mathcal{M}}\{CIV(m) : CIV(m) \in \mathbb{R}^+\} \ , \tag{13}$$

where, by definition, $\arg\inf(\emptyset) := \infty$.

The block size m_{MCIV} minimizes the empirical variance of the upper bound in a subsampling confidence interval with nominal confidence level t. Using Theorem 2, the formula for the breakdown point of m_{MCIV} is given in the next corollary.

Corollary 6 *Let b be the breakdown point of estimator $\hat{\theta}_n$. For given $t \in (0,1)$, let $b_t^{OS}(m)$ be the overlapping subsampling upper $t-$quantile breakdown point in Theorem 2, as a function of the block size $m \in \mathcal{M}$. It then follows:*

$$b_t^{OS,MCIV} \leq \sup_{m\in\mathcal{M}} \inf_{j\in\{-k,..,k\}} b_t^{OS}(m+j). \tag{14}$$

The dependence of the breakdown point formula for the MCIV on the breakdown point of subsampling quantiles is identical to the iid case. However, the much smaller quantile breakdown points in the time series case make the data driven choice m_{MCIV} very unreliable in presence of outliers. For instance, for the block size $n = 120$ and a maximal breakdown point statistic such that $b = 0.5$, the breakdown point of MCIV for $t = 0.95$ is less than 0.05, i.e., just 6 outliers are sufficient to break down the MCIV data driven choice of m. For the same sample size, the breakdown point of the MCIV method is larger than 0.3 in the iid case.

4.1.2 Calibration Method

Another consistent method for a data driven choice of the block size m can be based on a calibration procedure in the spirit of Loh (1987). Again, we present this method for the case of one–sided confidence intervals only. The modifications for two-sided confidence intervals are straightforward.

Definition 7 *Fix $t \in (0,1)$ and let $(X_1^*, \ldots, X_n^*)$ be a nonoverlapping moving block bootstrap sample generated from $X_{(n)}$ with block size m. For each bootstrap sample, denote by $Q_t^{**}(m)$ the $t-$subsampling quantile according to block size m. The data driven block size according to the calibration method is defined by*

$$m_{CM} := \arg \inf_{m \in \mathcal{M}} \{ |t - P^* \left[\hat{\theta}_n \leq Q_t^{**}(m) \right] | : P^* \left[Q_t^{**}(m) \in \mathbb{R} \right] > 1 - t \}, \tag{15}$$

where, by definition, $\arg\inf(\emptyset) := \infty$, and P^* is the nonoverlapping moving block bootstrap probability distribution.

In the approximation of the unknown underlying data generating mechanism in Definition 7, we use a nonoverlapping moving block bootstrap for ease of exposition. It is possible to consider also other resampling methods; see, e.g., Romano and Wolf (2001). By definition, m_{CM} is the block size for which the bootstrap probability of the event $[\hat{\theta}_n \leq Q_t^{**}(m)]$ is as near as possible to the nominal level t of the confidence interval, but which at the same time ensures that the resampling quantile breakdown probability of the calibration method is less than t. The last condition is necessary to ensure that the calibrated block size m_{CM} does not imply a degenerate subsampling quantile $Q_t^{**}(m_{CM})$ with a too large probability.

Corollary 8 *Let b be the breakdown point of estimator $\hat{\theta}_n$, $t \in (0,1)$, and define:*

$$b_t^{OS**}(m) \leq \frac{1}{n} \cdot \left[\inf_{q \in \mathbb{N}, q \leq r} \left\{ p = \lceil mb \rceil \cdot q \middle| P\left(BIN\left(r, \frac{q}{r}\right) < \lceil Q^{OS} \rceil \right) < 1 - t \right\} \right],$$

where $Q^{OS} := \frac{\lceil (n-m+1)(1-t) \rceil + \lceil mb \rceil - 1}{m}$. It then follows:

$$b_t^{OS,CM} \leq \sup_{m \in \mathcal{M}} \{ b_t^{OS**}(m) \}. \tag{16}$$

Because of the use of the moving block bootstrap instead of the standard iid bootstrap in the CM for time series, Equation (16) is quite different from the formula for the iid case in Camponovo, Scaillet and Trojani (2010). Similar to the iid case, the theoretical results in Table 2 and the Monte Carlo results in the last section of this paper indicate a higher stability and robustness of the CM relative to the MCIV method. Therefore, from a robustness perspective, the former should be preferred when consistent bootstrap methods are available. As discussed in Romano and Wolf (2001), the application of the calibration method in some settings can be computationally expensive. In contrast to our fast robust resampling approach, a direct application of the subsampling to robust estimators can easily become computationally prohibitive in combination with the CM.

4.2 Moving Block Bootstrap

The data driven method for the block size selection in Hall, Horowitz and Jing (1995) first computes the optimal block size for a subsample of size $m < n$. In a second step it uses Richardson extrapolation in order to determine the optimal block size for the whole sample.

Definition 9 *Let $m < n$ be fixed and split the original sample in $n-m+1$ overlapping blocks of size m. Fix $l_{\min} < l_{\max} < m$ and for $l \in \{l_{\min}, .., l_{\max}\}$ denote by $Q_t^*(m,l,i)$ the $t-$moving block bootstrap quantile computed with the block size l using the bootstrap m-block $(X_i, \dots, X_{i+m-1})$, $1 \le i \le n-m+1$. $\overline{Q}_t^*(m,l) := \frac{1}{n-m+1}\sum_{i=1}^{i=n-m+1} Q_t^*(m,l,i)$ is the corresponding average quantile. Finally, denote by $Q_t^*(n,l')$ the $t-$moving block bootstrap quantile computed with block size $l' < n$ based on the original sample $X_{(n)}$. For $l \in \{l_{\min}, .., l_{\max}\}$ define the MSE index is defined as*

$$MSE(l) := \left(\overline{Q}_t^*(m,l) - Q_t^*(n,l')\right)^2 + \frac{1}{n-m+1}\sum_{i=1}^{n-m+1}\left(Q_t^*(m,l,i) - Q_t^*(n,l')\right)^2, \tag{17}$$

and set:

$$l_{HHJ} := \arg\inf_{l\in\{l_{\min},..,l_{\max}\}}\{MSE(l) : MSE(l) \in \mathbb{R}^+\} , \tag{18}$$

where, by definition, $\arg\inf(\emptyset) := \infty$. The optimal block size for the whole n-sample is defined by

$$m_{HHJ} := l_{HHJ}\left(\frac{n}{m}\right)^{1/5}. \tag{19}$$

As discussed in Bühlmann and Künsch (1999), the HHJ method is not fully data driven, because it is based on some starting parameter values m and l'. However, the algorithm can be iterated. After computing the first value m_{HHJ}, we can set $l' = m_{HHJ}$ and iterate the same procedure. As pointed out in Hall, Horowitz and Jing (1995), this procedure often converges in one step. Also for this data-driven method, the application of the classical bootstrap approach to robust estimators easily becomes computationally unfeasible.

Corollary 10 *Let b be the breakdown point of estimator $\hat{\theta}_n$. For given $t \in (0,1)$, let $b_t^{NB,m}(l)$ and $b_t^{OB,m}(l)$ be the nonoverlapping and overlapping moving block upper $t-$quantile breakdown point in Theorem 2, as a function of the block size $l \in \{l_{\min}, .., l_{\max}\}$ and a block size m of the initial sample. It then follows for $K = NS, OS$:*

$$b_t^{K,HHJ} \leq \frac{m}{n} \cdot \sup_{l \in \{l_{\min},..,l_{\max}\}} b_t^{K,m}(l). \tag{20}$$

The computation of the optimal block size l_{HHJ} based on smaller subsamples of size $l << m < n$, causes a large instability in the computation of m_{HHJ}. Because of this effect, the MSE index in (17) can easily deteriorate even with a small contamination. Indeed, it is enough that the computation of the quantile degenerates just in a single m-block in order to imply a degenerated MSE. Table 2 confirms this intuition. For $n = 120$, $b = 0.5$, and $t = 0.95$, the upper bound on the breakdown point of the HHJ method is half that of the CM, even if for small block sizes the quantile breakdown point of subsampling procedures is typically lower than that of bootstrap methods.

5 Monte Carlo Simulations and Empirical Application

We compare through Monte Carlo simulations the accuracy of classical resampling procedures and our fast robust approach in estimating the confidence interval of the autoregressive parameter in a linear AR(1). Moreover, as a final exercise, we consider an application to real data testing the predictability of future stock returns with the classic and our robust fast subsampling.

5.1 AR(1) Model

Consider the linear AR(1) model of the form:

$$X_t = \theta X_{t-1} + \epsilon_t, \tag{21}$$

where $t = 1, \ldots, n$, $X_0 = 0$, $|\theta| < 1$, and $\{\epsilon_t\}$ is a sequence of iid standard normal innovations. We denote by $\hat{\theta}_n^{OLS}$ the (nonrobust) OLS estimator of θ_0, which is the solution of equation

$$\psi_n^{OLS}(X_{(n)}, \hat{\theta}_n^{OLS}) := \frac{1}{n-1} \sum_{t=1}^{n-1} X_t (X_{t+1} - \hat{\theta}_n^{OLS} X_t) = 0. \tag{22}$$

To apply our robust fast resampling approach, we consider a robust estimator $\hat{\theta}_n^{ROB}$ defined by

$$\psi_n^{ROB}(X_{(n)}, \hat{\theta}_n^{ROB}) := \frac{1}{n-1} \sum_{t=1}^{n-1} h_c(X_t (X_{t+1} - \hat{\theta}_n^{ROB} X_t)) = 0, \tag{23}$$

where $h_c(x) := x \cdot min(1, c/|x|)$, $c > 1$, is the Huber function; see Künsch (1984).

To study the robustness of the different resampling methods under investigation we consider replacement outliers random samples $(\tilde{X}_1, \ldots, \tilde{X}_n)$ generated according to

$$\tilde{X}_t = (1 - p_t) X_t + p_t \cdot \check{X}, \tag{24}$$

where $\check{X} = C \cdot \max(X_1, \ldots, X_n)$ and p_t is an iid $0-1$ random sequence, independent of process (21) and such that $P[p_t = 1] = \eta$; see Martin and Yohai (1986). In the following experiments, we consider $C = 2$, while the probability of contamination is set to $\eta = 1\%$, which is a very small contamination of the original sample.

5.1.1 The Standard Strictly Stationary Case

We construct symmetric resampling confidence intervals for the true parameter θ_0. Hall (1988) and more recent contributions, as for instance Politis, Romano and Wolf (1999), highlight a better accuracy of symmetric confidence intervals, which even in asymmetric settings can be shorter than asymmetric confidence intervals. Andrews and Guggenberger (2009, 2010a) and Mikusheva (2007) also show that because of a lack of uniformity in pointwise asymptotics, nonsymmetric subsampling confidence intervals for autoregressive models can imply a distorted asymptotic size, which is instead correct for symmetric confidence intervals.

Using OLS estimator (22), we compute both overlapping subsampling and moving block bootstrap distributions for $\sqrt{n}|\hat{\theta}_n^{OLS} - \theta_0|$. Using robust estimator (23), we compute overlapping robust fast subsampling and moving block bootstrap distributions for $\sqrt{n}|\hat{\theta}_n^{ROB} - \theta_0|$. Standard resampling methods combined with data driven block size selection methods for robust estimator (23) are computationally too expensive.

We generate $N = 1000$ samples of size $n = 240$ according to model (21) for the parameter choices $\theta_0 = 0.5, 0.6, 0.7, 0.8$, and simulate contaminated samples $(\tilde{X}_1, \ldots, \tilde{X}_n)$ according to (24). We select the subsampling block size using MCIV and CM for $\mathcal{M} = \{8, 10, 12, 15\}$. For the bootstrap, we apply HHJ method with $l' = 12$, $m = 30$, $l_{min} = 6$, and $l_{max} = 10$. The degree of robustness is $c = 5$; the significance level is $1 - \alpha = 0.95$.

We first analyze the finite sample coverage and the power of resampling procedures in a test of the null hypothesis $\mathcal{H}_0 : \theta_0 = 0.5$. Figure 1 plots the empirical frequencies of rejection of the null hypothesis $\mathcal{H}_0 : \theta_0 = 0.5$, for $\theta_0 = 0.5$ and different values of the alternative hypothesis: $\theta_0 \in \{0.6, 0.7, 0.8\}$.

Without contamination (left column, $\eta = 0\%$), we find that our robust fast approach and the classical procedures provide accurate and comparable results. In particular, when $\theta_0 = 0.5$, the size values for the classical moving block bootstrap and subsampling with CM are 0.054 and 0.044, respectively. With our robust approach, for the robust fast bootstrap and robust fast subsampling with CM we obtain 0.058 and 0.056, which both imply size values very close to the nominal level $\alpha = 0.05$. When $\theta_0 \neq 0.5$, the proportion of rejections of our robust fast approach remains slightly larger than that of the classical methods. For instance, when $\theta_0 = 0.7$, this difference in power between robust fast subsampling and subsampling with MCIV is close to 0.05.

If we consider the contaminated Monte Carlo simulations (right column, $\eta = 1\%$), the size increases for $\theta_0 = 0.5$ for nonrobust methods, which are found to be dramatically oversized. In the case of nonrobust subsampling methods the size is even larger than 0.4. In contrast, the size of our robust fast approach remains closer to the nominal level $\alpha = 0.05$. In particular, the size is 0.061 for the robust fast subsampling with CM. A contamination tremendously deteriorates also the power of nonrobust methods. As θ_0 increases, we find that the power curve of nonrobust methods is not monotonically increasing, with low frequencies of rejection even when θ_0 is far from 0.5. For instance, for $\theta_0 = 0.8$, the power of nonrobust moving block bootstrap method is smaller than 0.4, but that of our robust approach remains larger than 0.99.

In a second exercise, we examine the sensitivity of the different resampling procedures with respect to a single point contamination of the original sample. For each Monte Carlo sample, let:

$$X_{\max} = \arg \max_{X_1,\ldots,X_n} \{u(X_i)|u(X_i) = X_i - \theta X_{i-1}, \text{under}\, \mathcal{H}_0\}. \tag{25}$$

We modify $X_{\max}$ over a grid

$$\{X_{\max} + i, \quad i = 0, 1, 2, 3, 4, 5\} \tag{26}$$

Then, we analyze the sensitivity of the resulting empirical averages of p-values for testing the null hypothesis $\mathcal{H}_0 : \theta_0 = 0.5$. For this exercise the sample size is $n = 120$. In Figure 2, we plot the resulting empirical p-values. As expected, our robust fast approach shows a desirable stability for both subsampling and bootstrap methods.

5.1.2 The Near-to-Unit-Root Case

After the standard stationary case, we consider the near-to-unit root case. Since the limit distribution of the OLS estimator defined in Equation (22) is not continuous in parameter θ for $\theta_0 = 1$, moving block bootstrap procedures are inconsistent, see e.g., Basawa, Mallik, McCornick, Reeves and Taylor (1991). Recently, Phillips and Han (2008) and Han, Phillips and Sul (2011) have introduced new estimators for the autoregressive coefficient. Their limit distributions are normal and continuous in the interval $(-1, 1]$. Consequently, it makes sense to analyze the robustness properties of moving block bootstrap methods applied to these estimators in the near-to-unit root case. We also introduce a robust estimator based on the approach described in Han, Phillips and Sul (2011), and compare subsampling and moving block procedures to approximate their distributions.

Han, Phillips and Sul (2011) show that for $\theta_0 \in (-1, 1]$ and $s \leq t-3$, the following moment conditions hold:

$$E[(X_t - X_s)(X_{t-1} - X_{s+1}) - \theta_0(X_{t-1} - X_{s+1})^2] = 0. \tag{27}$$

Consequently, using Equations (27), they introduce (i) a method of moment estimator based on a single moment condition, i.e., s fixed, (ii) a GMM estimator (partially aggregated moment condition estimator) based on $L > 1$ moment conditions, with $L/n \to 0$, and (iii) a GMM estimator (fully aggregated moment condition estimator) based on all possible moment conditions.

In our exercise, we set $s = t - 3$ and consider the method of moment estimator $\hat{\theta}_n^M$ defined as the solution of $\psi_n^M(X_{(n)}, \hat{\theta}_n^M) = 0$, where

$$\psi_n^M(X_{(n)}, \theta) = \frac{1}{n-3} \sum_{t=4}^{n} (X_t - X_{t-3})(X_{t-1} - X_{t-2}) - \theta(X_{t-1} - X_{t-2})^2. \tag{28}$$

Given moment conditions (27), the following moment conditions can be defined, using Huber function h_c,

$$E\left[h_c\left((X_t - X_s)(X_{t-1} - X_{s+1}) - \theta_0(X_{t-1} - X_{s+1})^2\right)\right] = 0. \tag{29}$$

Equation (29) allows us to introduce the robust method of moment estimator $\hat{\theta}_n^{M,ROB}$ as the solution of $\psi_n^{M,ROB}(X_{(n)}, \hat{\theta}_n^{M,ROB}) = 0$, where

$$\psi_n^{M,ROB}(X_{(n)}, \theta) = \frac{1}{n-3} \sum_{t=4}^{n} h_c\left((X_t - X_{t-3})(X_{t-1} - X_{t-2}) - \theta(X_{t-1} - X_{t-2})^2\right). \tag{30}$$

Using the same arguments as in Han, Phillips and Sul (2011), the asymptotic distribution of the robust estimator defined in (30) is normal and continuous in the interval $(-1, 1]$, so that moving block bootstrap methods are consistent. Therefore, we can compare both overlapping subsampling and moving block bootstrap distributions for $\sqrt{n}|\hat{\theta}_n^M - \theta_0|$, and overlapping robust fast subsampling and moving block bootstrap distributions for $\sqrt{n}|\hat{\theta}_n^{M,ROB} - \theta_0|$.

We generate N=1000 samples of size $n = 240$ according to Model (21) for the parameter choices $\theta_0 = 0.7, 0.8, 0.9, 0.99$, and single point contamination of the original samples as in (26) with $i = 3$. We select the subsampling block size using MCIV and CM for $M = \{8, 10, 12, 15\}$. For the bootstrap, we apply HHJ method with $l' = 12$, $m = 30$, $l_{min} = 6$, and $l_{max} = 10$. The degree of robustness is $c = 3$.

We analyze the finite sample size and the power of resampling procedures in a test of the null hypothesis $\mathcal{H}_0 : \theta_0 = 0.7$. The significance level is $1-\alpha = 0.95$. Figure 3 plots the empirical frequencies of rejection of the null hypothesis $\mathcal{H}_0 : \theta_0 = 0.7$, for $\theta_0 = 0.7$, and different values $\theta_0 \in \{0.8, 0.9, 0.99\}$ of the alternative hypothesis.

As in the previous Monte Carlo setting, we find that without contamination (left column) our robust fast approach and the classical procedures yield accurate and comparable results. When $\theta_0 = 0.7$, the difference between the nominal level $\alpha = 0.05$ and the size of all methods under investigation is less than 0.016. For large θ_0, the power of nonrobust and robust methods is very similar. It is very interesting to note how the presence of a single outlier can deteriorate the accuracy of classical procedures (right column). For $\theta_0 = 0.7$ the size of the robust approach remains very close to the nominal level. In contrast, the size of nonrobust methods is close to 0.1. More strikingly, we also find that a single point contamination tremendously deteriorates the power of classical procedures. As θ_0 increases towards the boundary value 1, the frequencies of rejection of nonrobust methods are less than 30% even when $\theta_0 = 0.99$. In contrast, the power of the robust approach is substantial and larger than 55% for $\theta_0 = 0.99$.

5.2 Stock Return Predictability

Consider the predictive regression model:

$$y_t = \alpha + \beta z_{t-1} + \epsilon_t, \tag{31}$$

where, for $t = 1, \ldots, n$, $\{y_t\}$ denotes the stock return, $\{z_t\}$ denotes the explanatory variable and $\{\epsilon_t\}$ is the error term. We use the subscript 0 to indicate the true value β_0 of the parameter β.

Recently, several testing procedures have been proposed in order to test the nonpredictability hypothesis $\mathcal{H}_0 : \beta_0 = 0$; see among others Campbell and Yogo (2006), Jansson and Moreira (2006), and Amihud, Hurvich and Wang (2008). Indeed, because of the endogeneity of the explanatory variables in this setting, classic asymptotic theory based on OLS estimator becomes inaccurate. Moreover, as emphasized in Torous, Valkanov and Yan (2004), various state variables considered as predictors follows a nearly integrated process, which complicates inference on parameter β. As advocated, e.g., in Wolf (2000), the subsampling approach can be applied for testing the hypothesis of no predictability.

In this study, we analyze the predictive power of dividend yields for stock returns with the classic subsampling and our robust approach. We define the one-period real total return as

$$R_t = (P_t + d_t)/P_{t-1}, \tag{32}$$

where P_t is the end of month real stock price and d_t is the real dividend paid during month t. Furthermore, we define the annualized dividend series D_t as

$$D_t = d_t + (1 + r_t)d_{t-1} + (1 + r_t)(1 + r_{t-1})d_{t-2} + \cdots + (1 + r_t) \ldots (1 + r_{t-10})d_{t-11}, \tag{33}$$

where r_t is the one-month treasury-bill rate. Finally, we set $y_t = \ln(R_t)$ and $z_t = D_t/P_t$.

We compute the classic subsampling and our robust fast subsampling distributions for $\sqrt{n}|\hat{\beta}_n^{OLS} - \beta_0|$ and $\sqrt{n}|\hat{\beta}_n^{ROB} - \beta_0|$, respectively, where $\hat{\beta}_n^{OLS}$ and $\hat{\beta}_n^{ROB}$ are defined as the solutions of $\psi_n^{OLS}(Z_{(n)}, \hat{\beta}_n^{OLS}) = 0$ and $\psi_n^{ROB}(X_{(n)}, \hat{\beta}_n^{ROB}) = 0$, with $X_{(n)} = ((z_0, y_1), \ldots, (z_{n-1}, y_n))$ and

$$\psi_n^{OLS}(X_{(n)}, \hat{\theta}_n^{OLS}) \;:=\; \frac{1}{n-1}\sum_{t=1}^{n-1} z_{t-1}(y_t - \hat{\theta}_n^{OLS} z_{t-1}) = 0, \tag{34}$$

$$\psi_n^{ROB}(X_{(n)}, \hat{\theta}_n^{ROB}) \;:=\; \frac{1}{n-1}\sum_{t=1}^{n-1} h_c(z_{t-1}(y_t - \hat{\theta}_n^{ROB} z_{t-1})) = 0. \tag{35}$$

We consider monthly S&P 500 index data from Shiller (2000). In Figure 4 and Figure 5 we plot the log of return and the dividend yield of the S&P 500 for the period 1979-2009. The nearly integrated features of the dividend yield are well-known and apparent in Figure 5. We test the nonpredictability hypothesis $\mathcal{H}_0 : \beta_0 = 0$ for this period, consisting of 360 observations. To this end, we construct the classic and robust 95% subsampling confidence interval I_{CS} and I_{RS}, respectively, for parameter β. Based on these observations we obtain

$$I_{CS} \;=\; [-0.1518\,; 0.2271], \tag{36}$$

$$I_{RS} \;=\; [0.0225\,; 0.0629]. \tag{37}$$

It is interesting to note that in contrast to the classic subsampling our robust approach produces significant evidence of predictability. Indeed, the classic subsampling implies an extremely large confidence interval which lead to a nonrejection of $\mathcal{H}_0$. This finding seems to confirm the robustness problem of the classic approach in our Monte Carlo simulations. More precisely, Figure 6 plots the Huber weights

$$w_i := \min(1; c/\|z_{t-1}(y_t - \hat{\theta}_n^{ROB} z_{t-1})\|), \quad i = 1, \ldots, 360, \tag{38}$$

for the period 1979-2009, and clearly point out the presence of a large proportion of anomalous observations (z_{i-1}, y_i) with $\|z_{t-1}(y_t - \hat{\theta}_n^{ROB} z_{t-1})\| > c$. As shown in the previous section, the presence of anomalous observations may dramatically deteriorate the performance of nonrobust resampling methods. Consequently, the nonrejection of $\mathcal{H}_0$ caused by the large confidence interval provided by the classic subsampling suggests a low power of this approach in this case.

6 Conclusions

Theoretical breakdown point formulas and Monte Carlo evidence highlight a dramatic unexpected lack of robustness of classical block resampling methods for time series. This problem affects block bootstrap and subsampling procedures as well, and it is much worse than a related problem analyzed recently by the literature in the iid context. To overcome the problem, we propose a general robust fast resampling approach, which is applicable to a wide class of block resampling methods, and show that it implies good theoretical quantile breakdown point properties. In the context of a simple linear AR(1) model, our Monte Carlo simulations show that the robust resampling delivers more accurate and efficient results, in some cases to a dramatic degree, than other standard block resampling schemes in presence and absence of outliers in the original data. A real data application to testing for stock returns predictability provides more consistent evidence in favor of the predictability hypothesis using our robust resampling approach.

Appendix: Proofs

Proof of Theorem 2. The value $\frac{\lceil mb \rceil}{n}$ is the smallest fraction of outliers, that causes the breakdown of statistic T in a block of size m. Therefore, the first inequality is satisfied.

For the second inequality, we denote by $X^N_{(m),i} = (X_{(i-1)m+1}, \dots, X_{im})$, $i = 1, \dots, r$ and $X^O_{(m),i} = (X_i, \dots, X_{i+m-1})$, $i = 1, \dots, n-m+1$, the nonoverlapping and overlapping blocks of size m, respectively. Given the original sample $X_{(n)}$, for the first nonoverlapping block $X^N_{(m),1}$, consider the following type of contamination:

$$X^N_{(m),1} = (X_1, \dots, X_{m-\lceil mb \rceil}, Z_{m-\lceil mb \rceil+1}, \dots, Z_m), \tag{39}$$

where X_i, $i = 1, \dots, m - \lceil mb \rceil$ and Z_j, $j = m - \lceil mb \rceil + 1, \dots, m$, denote the noncontaminated and contaminated points, respectively. By construction, the first $m - \lceil mb \rceil + 1$ overlapping blocks $X^O_{(m),i}$, $i = 1, \dots, m - \lceil mb \rceil + 1$, contain $\lceil mb \rceil$ outliers. Consequently, $T(X^O_{(m),i}) = +\infty$, $i = 1, \dots, m - \lceil mb \rceil + 1$. Assume that the first $p < r - 1$ nonoverlapping blocks $X^N_{(m),i}$, $i = 1, \dots, p$, have the same contamination as in (39). Because of this contamination, the number of statistics $T^{OS*}_{n,m}$ which diverge to infinity is $mp - \lceil mb \rceil + 1$.

$Q^{OS*}_{t,n,m} = +\infty$, when the proportion of statistics $T^{OS*}_{n,m}$ with $T^{OS*}_{n,m} = +\infty$ is larger than $(1-t)$. Therefore, $b^{OS}_{t,n,m} \leq \inf_{\{p \in \mathbb{N}, p \leq r-1\}} \left\{ p \cdot \frac{\lceil mb \rceil}{n} \,\middle|\, \frac{mp - \lceil mb \rceil + 1}{n-m+1} > 1 - t \right\}$. ■

Proof of Theorem 3. The proof of the first inequalities follows the same lines as the proof of the first inequality in Theorem 2. We focus on the second inequalities.

Case (i): Nonoverlapping Moving Block Bootstrap. Consider $X^N_{(m),i}$, $i = 1, \dots, r$. Assume that p_2 of these nonoverlapping blocks are contaminated with exactly p_1 outliers for each block, while the remaining $(r - p_2)$ are noncontaminated (0 outlier), where $p_1, p_2 \in \mathbb{N}$ and $p_1 \leq m$, $p_2 \leq r - 1$. The nonoverlapping moving block bootstrap constructs a n-sample randomly, by selecting with replacement r nonoverlapping blocks. Let X be the random number of contaminated blocks in the random bootstrap sample. It follows that $X \sim BIN(r, \frac{p_2}{r})$. By

Definition 1, $Q^{NB*}_{t,n,m} = +\infty$, when the proportion of statistics $T^{NB*}_{n,m}$ such that $T^{NB*}_{n,m} = +\infty$ is larger than $1-t$. The smallest number of outliers such that $T^{NB*}_{n,m} = +\infty$ is by definition nb. Let $p_1, p_2 \in \mathbb{N}, p_1 \leq m, p_2 \leq r-1$. Consequently,
$b^{NB}_{t,n,m} \leq \frac{1}{n} \cdot \left[\inf_{\{p_1,p_2 \in \mathbb{N}\}} \left\{ p = p_1 \cdot p_2 \middle| P\left(BIN\left(r, \frac{p_2}{r} \right) \geq \lceil \frac{nb}{p_1} \rceil \right) > 1-t \right\} \right]$.

Case (ii): Overlapping Moving Block Bootstrap. Given the original sample $X_{(n)}$, consider the same nonoverlapping blocks as in (i), where the contamination of the p_2 contaminated blocks has the structure defined in (39). The overlapping moving block bootstrap constructs a n-sample randomly selecting with replacement r overlapping blocks of size m. Let X be the random variable which denotes the number of contaminated blocks in the random bootstrap sample. It follows that $X \sim BIN(r, \frac{mp_2 - p_1 + 1}{n-m+1})$.

By Definition 1, $Q^{OB*}_{t,n,m} = +\infty$, when the proportion of statistics $T^{OB*}_{n,m}$ with $T^{OB*}_{n,m} = +\infty$ is larger than $(1-t)$. The smallest number of outliers such that $T^{OB*}_{n,m} = +\infty$ is by definition nb. Let $p_1, p_2 \in \mathbb{N}, p_1 \leq m, p_2 \leq r-1$. Consequently,
$b^{OB}_{t,n,m} \leq \frac{1}{n} \cdot \left[\inf_{\{p_1,p_2\}} \left\{ p = p_1 \cdot p_2 \middle| P\left(BIN\left(r, \frac{mp_2 - p_1 + 1}{n-m+1} \right) \geq \lceil \frac{nb}{p_1} \rceil \right) > 1-t \right\} \right]$. ■

Proof of Corollary 4. Consider the robust fast approximation of $\hat{\theta}^{K*}_{n,m} - \hat{\theta}_n$ given by:

$$-[\nabla_\theta \psi_n(X_{(n)}, \hat{\theta}_n)]^{-1} \psi_k(X^{K*}_{(n,m),s}, \hat{\theta}_n), \tag{40}$$

where $k = n$ or $k = m$, $K = OS, NB, OB$. Assuming a bounded estimating function, expression (40) may degenerate only when either (i) $\hat{\theta}_n \notin \mathbb{R}$ or (ii) matrix $[\nabla_\theta \psi_n(X_{(n)}, \hat{\theta}_n)]$ is singular, i.e., $det([\nabla_\theta \psi_n(X_{(n)}, \hat{\theta}_n)]) = 0$. If (i) and (ii) are not satisfied, then, the quantile $Q^{K*}_{t,n,m}$ is bounded, for all $t \in (0,1)$. Let b be the breakdown point of $\hat{\theta}_n$ and $b_{\nabla\psi}$ the smallest fraction of outliers in the original sample such that condition (ii) is satisfied. Then, the breakdown point of $Q^{K*}_{t,n,m}$ is $b^K_{t,n,m} = \min(b, b_{\nabla\psi})$. ■

Proof of Corollary 6. Denote by $b^{OS}_t(m)$ the overlapping subsampling quantile breakdown point based on blocks of size m. By definition, in order to get $m_{MCIV} = \infty$, we must have $CIV(m) = \infty$ for all $m \in \mathcal{M}$. Given $m \in \mathcal{M}$, $CIV(m) = \infty$ if and only if the fraction of outliers

p in the sample $X_{(n)}$ satisfies $p \geq \min\{b_t^{OS}(m-k), b_t^{OS}(m-k+1), .., b_t^{OS}(m+k-1), b_t^{OS}(m+k)\}$. This concludes the proof. ■

Proof of Corollary 8. By definition, in order to imply $m_{CM} = \infty$, we must have $P[Q_t^{**}(m) = \infty] \geq t$ for all $m \in \mathcal{M}$. Given the original sample, Assume that q nonoverlapping blocks are contaminated with exactly $\lceil mb \rceil$ outliers for each block, while the remaining $(r-q)$ are noncontaminated (0 outliers), where $q \in \mathbb{N}$ and $q \leq r$. Moreover, assume that the contamination of the contaminated blocks has the structure defined in (39). Let X be the random variable which denotes the number of contaminated blocks in the nonoverlapping moving block bootstrap sample. Then, $X \sim BIN(r, q/r)$. For the construction of the nonoverlapping moving block bootstrap sample, the selection of $p \leq r-1$ contaminated blocks implies the breakdown of $mp - \lceil mb \rceil + 1$ overlapping subsampling statistics.

$Q_t^{**}(m) = \infty$, when the proportion of contaminated blocks is larger than $1-t$, i.e. $\frac{mp-\lceil mb \rceil+1}{n-m+1} > 1-t \Leftrightarrow p > \frac{\lceil (n-m+1)(1-t) \rceil + \lceil mb \rceil - 1}{m}$. This concludes the proof of the second statement. ■

Proof of Corollary 10. By definition, in order to get $m_{HHJ} = \infty$, we must have $l_{HHJ} = \infty$, i.e., $MSE(l) = \infty$ for all $l \in \{l_{min} \dots, l_{max}\}$. For l fixed, $MSE(l) = \infty$ if just a single $Q_t^*(m, l, i)$, $i = 1, \dots, n-m+1$ diverges to infinity. This concludes the proof. ■

References

[1] Amihud, Y., Hurvich C.M., and Y. Wang, 2008. Multiple-predictor regressions: hypothesis testing. The Review of Financial Studies, 22, 413–434.

[2] Andrews, D., 2000. Inconsistency of the bootstrap when a parameter is on the boundary of the parameter space. Econometrica, 68, 399–405.

[3] Andrews, D., 2002. Higher-order improvements of a computationally attractive k-step bootstrap for extremum estimators. Econometrica, 70, 119–162.

[4] Andrews, D., 2004. The block-block bootstrap: improved asymptotic refinements. Econometrica, 72, 673–700.

[5] Andrews, D., and P. Guggenberger, 2009. Hybrid and size-corrected subsample methods. Econometrica, 77, 721–762.

[6] Andrews, D., and P. Guggenberger, 2010a. Asymptotic size and a problem with subsampling and with the m out of n bootstrap. Econometric Theory, 26, forthcoming.

[7] Andrews, D., and P. Guggenberger, 2010b. Application of subsampling, hybrid and size-correction methods. Journal of Econometrics, forthcoming.

[8] Basawa, I. V., Mallik, A. K., McCornick W. P., Reeves J. H., and R. L. Taylor, 1991. Bootstrapping unstable first-order autoregressive process. Annals of Statistics, 19, 1098–1101.

[9] Bickel, P. J., Gotze, F., and W. R. van Zwet, 1997. Resampling fewer than n observations: Gains, losses, and remedies for losses. Statistica Sinica, 7, 1–31.

[10] Bühlmann, P., 2002. Bootstraps for time series. Statistical Science, 17, 52-72.

[11] Bühlmann, P., and H. Künsch, 1999. Block length selection in the bootstrap for time series. Computational Statistics and Data Analysis, 31, 295-310.

[12] Campbell, J.Y., and M. Yogo, 2006. Efficient tests of stock return predictability. Journal of Financial Economics, 81, 27–60.

[13] Camponovo, Scaillet and Trojani, 2010. Robust Subsampling. Working Paper.

[14] Carlstein, E., 1986. The use of subseries methods for estimating the variance of a general statistic from a stationary time series. Annals of Statistics, 14, 1171–1179.

[15] Davidson, R., and J. McKinnon, 1999. Bootstrap testing in nonlinear models. International Economic Review, 40, 487–508.

[16] Efron, B., and R. Tibshirani, 1993. An Introduction to the Bootstrap. New York: Chapman and Hall.

[17] Genton, M., and A. Lucas, 2003. Comprehensive definitions of breakdown point for independent and dependent observations. Journal of the Royal Statistical Society, Series B, 65, 81–94.

[18] Goncalves, S., and H. White, 2004. Maximum likelihood and the bootstrap for nonlinear dynamic models. Journal of Econometrics, 119, 199–220.

[19] Hall, P., 1985. Resampling a coverage process. Stochastic Processes and their Applications, 19, 259–269.

[20] Hall, P., 1988. On symmetric bootstrap confidence intervals. Journal of the Royal Statistical Society, Ser.B, 50, 35–45.

[21] Hall, P., 1992. The Bootstrap and Edgeworth Expansion. New York: Springer-Verlag.

[22] Hall, P., and J. Horowitz, 1996. Bootstrap critical values for tests based on Generalized-Method-of-Moment estimators. Econometrica, 64, 891–916.

[23] Hall, P., Horowitz J., and B.-Y. Jing, 1995. On blocking rules for the bootstrap with dependent data. Biometrika, 82, 561–574.

[24] Hampel, F. R., Ronchetti, E. M., Rousseeuw, P . J., and W. A. Stahel, 1986. *Robust statistics. The approach based on influence functions*. Wiley, Mew York.

[25] Han, C., Phillips, P.C.B., and D. Sul, 2011. Uniform asymptotic normality in stationary and unit root autoregression. Econometric Theory, forthcoming.

[26] Hong, H., and O. Scaillet, 2006. A fast subsampling method for nonlinear dynamic models. Journal of Econometrics, 133, 557–578.

[27] Hu, F., and J. Kalbfleisch, 2000. The estimating function bootstrap. Canadian Journal of Statistics, 28, 449–499

[28] Jansson, M., and M.J. Moreira, 2006. Optimal inference in regression models with nearly integrated regessors. Econometrica, 74, 681–714.

[29] Künsch, H., 1984. Infinitesimal robustness for autoregressive processes. Annals of Statistics, 12, 843–863.

[30] Künsch, H., 1989. The jacknife and the bootstrap for general stationary observations. Annals of Statistics, 17, 1217–1241.

[31] Lahiri, S.N., 2001. Effects of block lenghts on the validity of block resampling methods. Probability Theory and Related Fields, 121, 73–97.

[32] Lahiri, S.N., 2003. Resampling methods for dependent data. Springer, New York.

[33] La Vecchia, D., and F. Trojani, 2010. Infinitesimal robustness for diffusions. Journal of the American Statistical Association, forthcoming.

[34] Loh, W. Y., 1987. Calibrating confidence coefficients. Journal of the American Statistical Association, 82, 155–162.

[35] Mancini, L., Ronchetti, E., and F. Trojani, 2005. Optimal conditionally unbiased bounded-influence inference in dynamic location and scale models. Journal of the American Statistical Association, 100, 628–641.

[36] Martin, R.D., and V. Yohai, 1986. Influence functionals for time series. Annals of Statistics, 14, 781–818.

[37] Mikusheva, A., 2007. Uniform inference in autoregressive models. Econometrica, 75, 1411–1452.

[38] Muler, N., and V. Yohai, 2008. Robust estimates for GARCH models. Journal of Statistical Planning and Inference, 138, 2918–2940.

[39] Ortelli, C., and F. Trojani, 2005. Robust efficient method of moments. Journal of Econometrics, 128, 69–97.

[40] Phillips, P.C.B., and C. Han, 2008. Gaussian inference in AR(1) time series with or without a unit root. Econometric Theory, 24, 631-650.

[41] Politis, D. N., and J. P. Romano, 1992. A general theory for large sample confidence regions based on subsamples under minimal assumptions. Technical Report 399, Dept of Statistics, Stanford University.

[42] Politis, D. N., and J. P. Romano, 1994a. Large sample confidence regions based on subsamples under minimal assumptions. Annals of Statistics, 22, 203–2050.

[43] Politis, D. N., and J. P. Romano, 1994b. The stationary bootstrap. Journal of the American Statistical Association, 89, 1303–1313.

[44] Politis, D. N., Romano J. P., and M. Wolf, 1999. Subsampling. Springer, New York.

[45] Romano, J. P., and M. Wolf, 2001. Subsampling intervals in autoregressive models with linear time trend. Econometrica, 69, 1283-1314.

[46] Ronchetti, E., and F. Trojani, 2001. Robust inference with GMM estimators, Journal of econometrics, 101, 37–69.

[47] Sakata, S., and H. White, 1998. High breakdown point conditional dispersion estimation with application to S&P500 daily returns volatility. Econometrica, 66, 529–567.

[48] Salibian-Barrera, M., Van Aelst, S., and G. Willems, 2006. Principal components analysis based on multivariate MM estimators with fast and robust bootstrap. Journal of the American Statistical Association, 101, 1198–1211.

[49] Salibian-Barrera, M., Van Aelst, S., and G. Willems, 2007. Fast and robust bootstrap. Statistical Methods and Applications, 17, 41–71

[50] Salibian-Barrera, M., and R. Zamar, 2002. Boostrapping robust estimates of regression. Annals of Statistics, 30, No. 2, 556–582.

[51] Shao, J., and D. Tu, 1995. The jackknife and bootstrap. Springer, New York.

[52] Singh, K., 1998. Breakdown theory for bootstrap quantiles. Annals of Statistics, 26, No. 5, 1719–1732.

[53] Shiller, R.J., 2000. Irrational Exuberance. Princeton University Press, Princeton, New York.

[54] Torous, W., R. Valkanov, and S. Yan, 2004. On predicting stock returns with nearly integrated explanatory variables. Journal of Business, 77, 937-966.

[55] Wolf, M., 2000. Stock returns and dividend yields revisited: a new way to look at an old problem. Journal of Business and Economic Statistics, 18, 18-30.

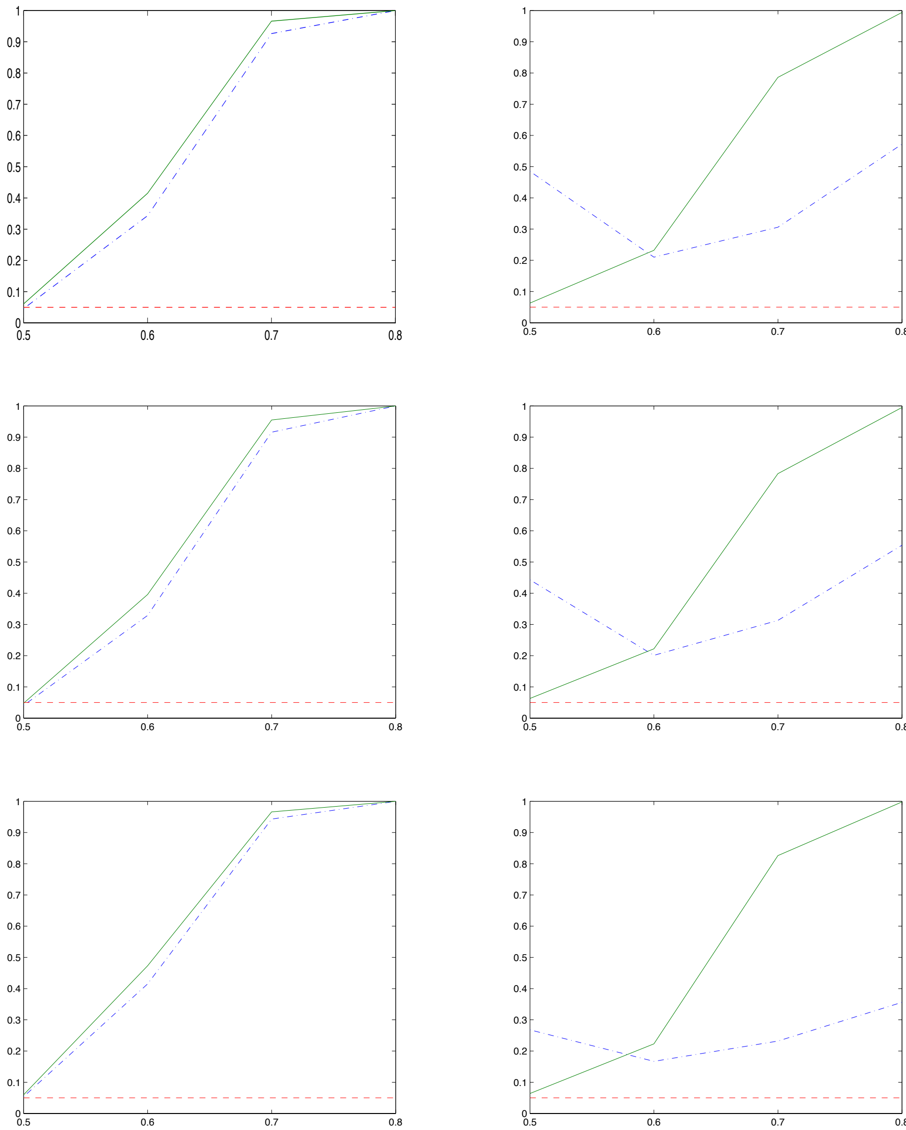

Figure 1: **Power curves in the standard stationary case.** We plot the proportion of rejections of the null hypothesis $\mathcal{H}_0 : \theta_0 = 0.5$, when the true parameter value is $\theta_0 \in [0.5, 0.8]$. From the top to the bottom, we present the overlapping subsampling with MCIV, the subsampling with CM and the moving block bootstrap with HHJ. We consider our robust fast approach (straight line) and the classic approach (dash-dotted line). In the left column, we consider a noncontaminated sample ($\eta = 0\%$). In the right column, the proportion of outliers is $\eta = 1\%$.

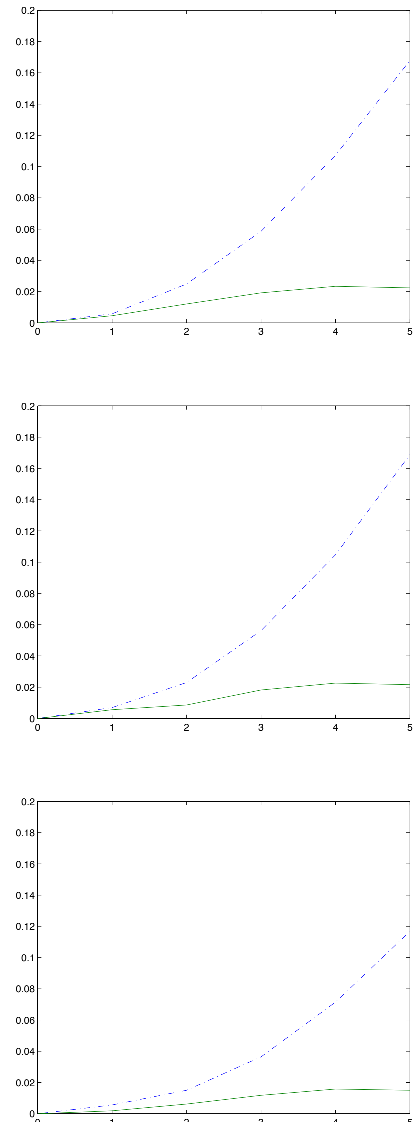

Figure 2: **Sensitivity analysis.** Sensitivity plots of the variation of the empirical $p-$value average, for a test of the null hypothesis $\mathcal{H}_0 : \theta_0 = 0.5$, with respect to variations of $X_{\max}$, in each Monte Carlo sample, within the interval $[0, 5]$. The random samples were generated under $\mathcal{H}_0$. From the top to the bottom, we present the overlapping subsampling with MCIV, the subsampling with CM and the moving block bootstrap with HHJ. We consider the robust fast approach (straight line) and the classic nonrobust approach (dash-dotted line).

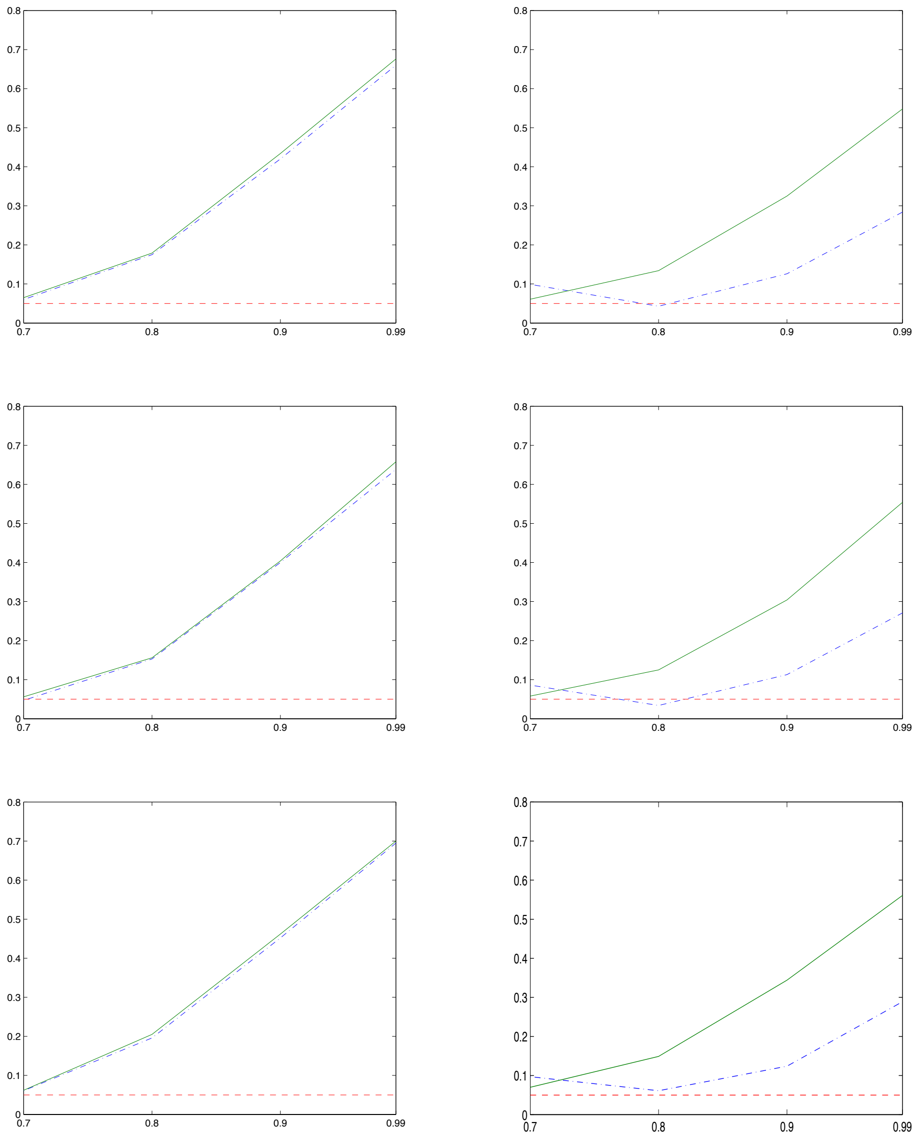

Figure 3: **Power curves in the near-to-unit-root case.** We plot the proportion of rejections of the null hypothesis $\mathcal{H}_0 : \theta_0 = 0.7$, when the true parameter value is $\theta_0 \in [0.7, 0.99]$. From the top to the bottom, we present the overlapping subsampling with MCIV, the subsampling with CM and the moving block bootstrap with HHJ. We consider our robust fast approach (straight line) and the classic approach (dash-dotted line). In the left column, we consider noncontaminated samples. In the right column, we consider single point contamination.

$n = 120$, $b = 0.5$	0.95	0.99
O. Subsampling ($m = 5$)	$[0.0250; 0.0500]$	$[0.0250; 0.0250]$
O. Subsampling ($m = 10$)	$[0.0417; 0.0417]$	$[0.0417; 0.0417]$
O. Subsampling ($m = 20$)	$[0.0833; 0.0833]$	$[0.0833; 0.0833]$
N. Bootstrap ($m = 5$)	$[0.0250; 0.3333]$	$[0.0250; 0.2917]$
N. Bootstrap ($m = 10$)	$[0.0417; 0.2500]$	$[0.0417; 0.2250]$
N. Bootstrap ($m = 20$)	$[0.0667; 0.1667]$	$[0.0667; 0.1667]$
O. Bootstrap ($m = 5$)	$[0.0250; 0.3750]$	$[0.0250; 0.2917]$
O. Bootstrap ($m = 10$)	$[0.0417; 0.3333]$	$[0.0417; 0.2500]$
O. Bootstrap ($m = 20$)	$[0.0667; 0.3000]$	$[0.0667; 0.2500]$

Table 1: **Subsampling and Moving Block Bootstrap Lower and Upper Bounds for the Quantile Breakdown Point.** Breakdown point of the overlapping (O.) subsampling and nonoverlapping (N.) and overlapping (O.) moving block bootstrap quantile. The sample size is $n = 120$, the block size $m = 5, 10, 20$. We assume a statistic with breakdown point $b = 0.5$ and confidence level $t = 0.95, 0.99$. Lower and upper bounds for quantile breakdown points are computed using Theorem 2 and 3.

$n = 120$	$t = 0.95$	$t = 0.99$
O. Subsampling MCIV	≤ 0.0500	≤ 0.0500
O. Subsampling CM	≤ 0.2000	≤ 0.2667
N. Bootstrap HHJ	≤ 0.1000	≤ 0.0667
O. Bootstrap HHJ	≤ 0.1000	≤ 0.0667

Table 2: **Breakdown point of Block Size Selection Procedures**. We compute the breakdown point of the minimum confidence index volatility (MCIV), the calibration method (CM), and the data driven method in Hall, Horowitz and Jing (1995) (HHJ) for the nonoverlapping (N.) and overlapping (O.) cases. For (MCIV) and (CM) we use Corollary 6, 8 with $\mathcal{M} = \{6, 8, 10, 12, 15\}$. For (HHJ) we use Corollary 10 with $m = 30$, $l_{min} = 3$, and $l_{\max} = 10$. The breakdown point of the statistic is $b = 0.5$ and the confidence levels are $t = 0.95, 0.99$. The sample size is $n = 120$.

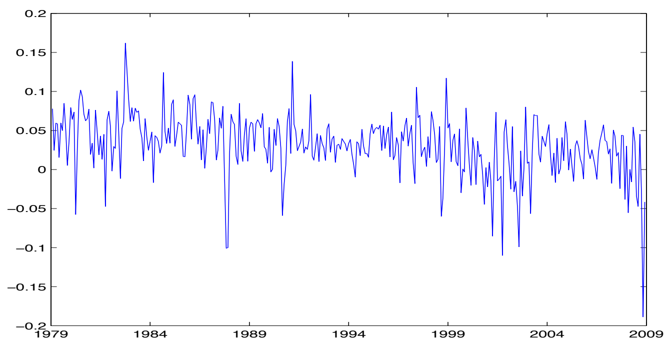

Figure 4: **Log of stock returns.** We plot the log of stock return of the S&P 500 for the period 1979-2009. We consider monthly S&P 500 index data from Shiller (2000).

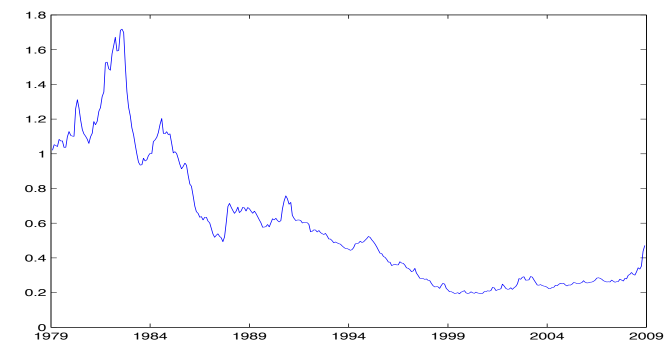

Figure 5: **Dividend yield.** We plot the dividend yield of the S&P 500 for the period 1979-2009. We consider monthly S&P 500 index data from Shiller (2000).

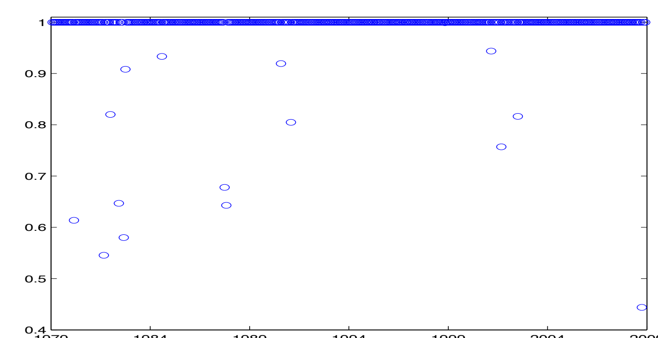

Figure 6: **Huber weights.** We plot the Huber weights $w_i := \min(1; c/\|z_{t-1}(y_t - \hat{\theta}_n^{ROB} z_{t-1})\|)$, $i = 1, \ldots, 360$, for the robust regression defined in Equation (35) for the period 1979-2009. We consider monthly S&P 500 index data from Shiller (2000).