The Entropy Method and Repeated Games with Bounded Memory

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- Bounded memory.

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Entropy

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- The mutual information of a pair of discrete random variables X and Y, denoted $I(X;Y)$, expresses the amount of information conveyed by X about Y and vice versa (since $I(X;Y) = I(Y;X)$). As such, it measured the amount of dependence between X and Y.

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- Player 1 plays a mixed *oblivious* strategy X.
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- The pair (X, τ) induces a random play $(x_1, y_1, \ldots, x_n, y_n)$, where $X = (x_1, \ldots, x_n)$ and $y_t = \tau(x_1, \ldots, x_{t-1})$.

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- We want to evaluate the expected average payoffs in the n fold repeated game. It is sufficient to consider the expected empirical frequency of the play, i.e. the expected number of times that each action profile was played divided by n.

Lemma (Neyman-Okada 2009)

Let X and τ be (correlated) random variables assuming values in

- X - A^n,
- τ - the set of pure strategies of player 2 in the n fold repeated version of G.

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Let a and b be random variables whose joint distribution is the expected empirical frequency of the induced play. We have

$$I(a; b) \leq H(a) - \frac{1}{n} H(X) + \frac{1}{n} I(X; \tau).$$

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- If X was a sequence of i.i.d. random variables and X and τ were independent, then the payoff of player 1 would be at least the payoff secured by a in one-stage game.
- In general, the payoff of player 1 is at least the payoff secured by a in the one-stage game where player 2 is allowed to correlate with player 1 up to the amount on the right-hand side of the inequality.

Uninformed game, revisited

$$I(a;b) \leq \left[\ H(a) \ - \ \frac{1}{n2^n} H(x_1, \ldots, x_{n2^n}) \ \right] + \ \frac{1}{n2^n} I(x_1, \ldots, x_{n2^n}; \tau)$$

- Player 2 is not informed of the realization of $x_1, \ldots, x_{n2^n}$.

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- Conclusion:

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- $\log m!/m\log m \to 1$
- Conclusion: the value of the uninformed game converges to the value of the one-stage game.

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- For every realization $\xi = \xi_1, \ldots, \xi_{n2^n}$, the player must specify a strategy in the n fold repeated matching pennies game, $\tau_{|\xi}$. The informed strategy $\tau : \xi \mapsto \tau_{|\xi}$ is a random variable that depends on $x_1, \ldots, x_{n2^n}$.

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- A player who can only memorize M bits of information, can only use informed strategies that have at most 2^M distinct realizations. Namely,

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- We use the above as a definition for M-memory informed strategies ("0-memory" = "uninformed").

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- The entropy if a random variable is bounded by the binary logarithm of the number of its possible values.

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- The entropy if a random variable is bounded by the binary logarithm of the number of its possible values.
- Let M be the memory capacity of player 2.

Informed game, revisited

$$I(a;b) \leq \left[H(a) - \frac{1}{n2^n} H(x_1,\ldots,x_{n2^n})\right] + \frac{1}{n2^n} I(x_1,\ldots,x_{n2^n};\tau)$$
$$\leq \mathrm{o}(1) + \frac{1}{n2^n} H(\tau) \leq \mathrm{o}(1) + \frac{M}{n2^n}$$

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- Therefore, the value of the informed game converges to the value of the one-stage game where player 1 plays $(\frac{1}{2}, \frac{1}{2})$ and player 2 choose a correlated strategy whose mutual information is bounded by Θ.

Repeated Games with Bounded Memory

- The information of a player in a repeated game is the history up to the current stage. Thereby, an M-memory strategy τ is defined by

$$\log_2 \# \left\{ \tau_{|h} : \text{“}h \text{ is a finite history”} \right\} \leq M,$$

where $\tau_{|h}$ is the strategy that τ induces on the sub-game that starts after the history h has been played.

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- What is $C(Q)$? This is an open problem!
- In an ongoing joint work with Olivier Gossner and Penelope Hernandez we were able to show that

$$\min_i \left(\frac{H(Q_i)}{|A_{-i}| - 1} \right) \leq C(Q) \leq \max_i \left(\frac{H(Q)}{|A_i| - 1} \right)$$

Further results

Ron Peretz.
Conceald correlation through bounded recall strategies.
International Journal of Game Theory (forthcoming).

Ron Peretz.
Learning cycle length through finite automata.
Mathematics of Operations Research (forthcoming).

Ron Peretz.
The strategic value of recall.
Games and Economic Behavior (published online).