The Entropy Method and Repeated Games with Bounded Memory

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October 27
Introductory example

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- Take a random ordering of the integers 0, 1, ..., 7.
- Write them in binary representation. We have a random sequence of 24 bits, $x_1, \ldots, x_{24}$. 

Bounded memory.

Let's play! 

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Shannon’s entropy of a discrete random variable $X$, denoted $H(X)$, expresses the expected number of bits of information one might need in order to communicate the value of $X$, or equivalently, the number of bits of information conveyed by $X$. 
Entropy

- **Shannon’s entropy** of a discrete random variable $X$, denoted $H(X)$, expresses the expected number of bits of information one might need in order to communicate the value of $X$, or equivalently, the number of bits of information conveyed by $X$.

- The **mutual information** of a pair of discrete random variables $X$ and $Y$, denoted $I(X;Y)$, expresses the amount of information conveyed by $X$ about $Y$ and vice versa (since $I(X;Y) = I(Y;X)$). As such, it measured the amount of dependence between $X$ and $Y$. 
Consider the $n$ fold repeated version of a finite two-person game $G = \langle A, B, g \rangle$. 
Why entropy?

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Why entropy?

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- Player 1 plays a mixed oblivious strategy $X$.
- Player 2, who has some information about the realization of $X$, plays a correlated (nonoblivious) strategy $\tau$.
- The pair $(X, \tau)$ induces a random play $(x_1, y_1, \ldots, x_n, y_n)$, where $X = (x_1, \ldots, x_n)$ and $y_t = \tau(x_1, \ldots, x_{t-1})$. 
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- We want to evaluate the expected average payoffs in the $n$ fold repeated game. It is sufficient to consider the expected empirical frequency of the play, i.e. the expected number of times that each action profile was played divided by $n$. 
Lemma (Neyman-Okada 2009)

Let $X$ and $\tau$ be (correlated) random variables assuming values in

- $X - A^n$,
- $\tau$ - the set of pure strategies of player 2 in the $n$ fold repeated version of $G$.

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Let $a$ and $b$ be random variables whose joint distribution is the expected empirical frequency of the induced play. We have

$$I(a; b) \leq H(a) - \frac{1}{n} H(X) + \frac{1}{n} I(X; \tau).$$
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dependency

randomness

mutual

information
Neyman-Okada’s lemma, continued

\[ I(a; b) \leq H(a) - \frac{1}{n} H(X) + \frac{1}{n} I(X; \tau) \]

deependency  randomness  mutual

If \( X \) was a sequence of i.i.d. random variables and \( X \) and \( \tau \) were independent, then the payoff of player 1 would be at least the payoff secured by \( a \) in one-stage game.

In general, the payoff of player 1 is at least the payoff secured by \( a \) in the one-stage game where player 2 is allowed to correlate with player 1 up to the amount on the right-hand side of the inequality.
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Uninformed game, revisited

\[ I(a; b) \leq \left[ H(a) - \frac{1}{n^{2n}} H(x_1, \ldots, x_{n^{2n}}) \right] + \frac{1}{n^{2n}} I(x_1, \ldots, x_{n^{2n}}; \tau) \]

- Player 2 is not informed of the realization of \( x_1, \ldots, x_{n^{2n}} \).
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\[ \log m! \approx n \log m \rightarrow 1 \]

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- The entropy of the uniform distribution on \( m \) objects is \( \log_2 m \).
Uninformed game, revisited

\begin{align*}
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& \frac{\log_2 2^n!}{n^{2n}} \rightarrow 0
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\[ \log_2 2^n! / m \log_2 m \rightarrow 1 \]
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\[ I(a; b) \leq \left[ H(a) - \frac{1}{n2^2} H(x_1, \ldots, x_n2^2) \right] + \frac{1}{n2^n} I(x_1, \ldots, x_n2^n; \tau) \]

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- \( \log m! / m \log m \to 1 \)
- Conclusion:
Uninformed game, revisited

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- \( \log m! / m \log m \to 1 \)
- Conclusion: the value of the uninformed game converges to the value of the one-stage game.
Informed strategies and bounded memory

- A strategy of an informed player depends on his information. In our example, the information is the realization of $x_1, \ldots, x_{n2^n}$.
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For every realization $\xi = \xi_1, \ldots, \xi_{n2^n}$, the player must specify a strategy in the $n$ fold repeated matching pennies game, $\tau|\xi$. The informed strategy $\tau : \xi \mapsto \tau|\xi$ is a random variable that depends on $x_1, \ldots, x_{n2^n}$. 

A player who can only memorize $M$ bits of information, can only use informed strategies that have at most $2^M$ distinct realizations. Namely, $\log_2 \#\{\tau|\xi : \xi \in \{0, 1\}^{n2^n}\} \leq M$. 

We use the above as a definition for $M$-memory informed strategies ("0-memory"="uninformed").
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For every realization $\xi = \xi_1, \ldots, \xi_{n2^n}$, the player must specify a strategy in the $n$ fold repeated matching pennies game, $\tau|_\xi$. The informed strategy $\tau : \xi \mapsto \tau|_\xi$ is a random variable that depends on $x_1, \ldots, x_{n2^n}$.

A player who can only memorize $M$ bits of information, can only use informed strategies that have at most $2^M$ distinct realizations. Namely,

$$\log_2 \# \left\{ \tau|_\xi : \xi \in \{0, 1\}^{n2^n} \right\} \leq M.$$ 

We use the above as a definition for $M$-memory informed strategies (“0-memory” = “uninformed”).
Informed game, revisited

\[ I(a; b) \leq \left[ H(a) - \frac{1}{n2^n} H(x_1, \ldots, x_{n2^n}) \right] + \frac{1}{n2^n} I(x_1, \ldots, x_{n2^n}; \tau) \]
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- We have already seen that the first term on the right-hand side vanishes as \( n \) grows.
Informed game, revisited

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- We have already seen that the first term on the right-hand side vanishes as \( n \) grows.
- The amount of information that \( \tau \) conveys about \( x_1, \ldots, x_{n2^n}, \) \( I(x_1, \ldots, x_{n2^n}, \tau) \), is at most the total amount of information that \( \tau \) conveys, \( H(\tau) \).
\[ I(a; b) \leq \left[ H(a) - \frac{1}{n^{2n}} H(x_1, \ldots, x_{n^{2n}}) \right] + \frac{1}{n^{2n}} I(x_1, \ldots, x_{n^{2n}}; \tau) \]
\[ \leq o(1) + \frac{1}{n^{2n}} H(\tau) \]

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- The amount of information that \( \tau \) conveys about \( x_1, \ldots, x_{n2^n} \), \( I(x_1, \ldots, x_{n2^n}, \tau) \), is at most the total amount of information that \( \tau \) conveys, \( H(\tau) \).
- The entropy if a random variable is bounded by the binary logarithm of the number of its possible values.
\[ I(a; b) \leq \left[ H(a) - \frac{1}{n2^n} H(x_1, \ldots, x_{n2^n}) \right] + \frac{1}{n2^n} I(x_1, \ldots, x_{n2^n}; \tau) \]
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Informed game, revisited

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Informed game, conclusions

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- Therefore, the value of the informed game converges to the value of the one-stage game where player 1 plays \( \left(\frac{1}{2}, \frac{1}{2}\right) \) and player 2 choose a correlated strategy whose mutual information is bounded by \( \Theta \).
Repeated Games with Bounded Memory

- The information of a player in a repeated game is the history up to the current stage. Thereby, an $M$-memory strategy $\tau$ is defined by

$$\log_2 \# \{ \tau|_h : \text{"h is a finite history"} \} \leq M,$$

where $\tau|_h$ is the strategy that $\tau$ induces on the sub-game that starts after the history $h$ has been played.
Suppose two $M$-memory players correlate in order to produce an $n$-periodic sequence $x_1, x_2, \ldots$ that behaves similar to a sequence of i.i.d. random variables with distribution $Q \in \Delta(A_1 \times A_2)$. 

Let $C(Q)$ be the smallest real number such that, if $M^2 \geq C(Q)n$, then the above is possible.

What is $C(Q)$? This is an open problem!

In an ongoing joint work with Olivier Gossner and Penelope Hernandez we were able to show that

$$\min_i (H(Q_i) | A - i) - 1 \leq C(Q) \leq \max_i (H(Q_i) | A_i) - 1$$

Ron Peretz (ronprtz@gmail.com)
Correlation through bounded memory strategies

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- Namely, the expected empirical distribution is close to $Q$ and $H(x_1, \ldots, x_n)/n$ is close to $H(Q)$. 

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Entropy Method
October 27 12 / 13
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Further results

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The strategic value of recall. *Games and Economic Behavior* (published online).