Who's Who in Crime Networks. Wanted: The Key Player *

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Abstract

Criminals are embedded in a network of relationships. Social ties among criminals are modeled by means of a graph where criminals compete for a booty and benefit from local interactions with their neighbors. Each criminal decides in a non-cooperative way how much crime effort he will exert. We show that the Nash equilibrium crime effort of each individual is proportional to his *equilibrium Bonacich-centrality* in the network, thus establishing a bridge to the sociology literature on social networks. We then analyze a policy that consists of finding and getting rid of the *key player*, that is, the criminal who, once removed, leads to the maximum reduction in aggregate crime. We provide a geometric characterization of the key player identified with an *optimal inter-centrality measure*, which takes into account both a player's centrality and his contribution to the centrality of the others. We also provide a geometric characterization of the key group, which generalizes the key player for a group of criminals of a given size. We finally endogeneize the crime participation decision, resulting in a key player policy, which effectiveness depends on the *outside opportunities* available to criminals.

Keywords: Social networks, crime, centrality measures, key group, policies.

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1 Introduction

Polls show that people regard crime as the number one social problem. As such, identifying the root causes of criminal activity and designing efficient policies against crime are two natural scopes for the economics profession. About thirty years ago, the major breakthrough in the economic analysis of crime was the work of Gary Becker (1968) in which criminals are rational individuals acting in their own self-interest. In deciding to commit a crime, criminals weigh the expected costs against the expected benefits accruing from this activity. The goal of the criminal justice system is to raise expected costs of crime to criminals above the expected benefits. People will commit crimes only so long as they are willing to pay the prices society charges.

There is now a large literature on the economics of crime. Both theoretical and empirical approaches have been developed over the years in order to better understand the costs and benefits of crime (see, for instance, the literature surveys by Nuno Garoupa, 1997, and Mitchell A. Polinsky and Steven Shavell, 2000). In particular, the interaction between the "crime market" and the other markets has important general equilibrium effects that are crucial if one wants to implement the most effective policies.[1] The standard policy tool to reduce aggregate crime that is common to all these models relies on the deterrence effects of punishment, i.e., the planner should increase uniformly punishment costs.

It is however well-established that crime is, to some extent, a group phenomenon, and the source of crime and delinquency is located in the intimate social networks of individuals (see e.g. Edwin H. Sutherland, 1947, Jerzy Sarnecki, 2001 and Mark Warr, 2002). Indeed, criminals often have friends who have themselves committed several offences, and social ties among criminals are seen as a means whereby individuals exert an influence over one another to commit crimes. In fact, not only friends but also the *structure* of social networks matters in explaining individual's own criminal behavior. In adolescents' friendship networks, Dana L. Haynie (2001) shows that individual Bonacich centrality (a standard measure of network centrality) together with the density of friendship links condition the delinquency-peer association. This suggests that the underlying structural properties of friendship networks must be taken into account to better understand the impact of peer influence on criminal behavior and to address adequate and novel crime-reducing policies.

The aim of this paper is twofold. First, we relate individual crime outcomes to the agents' network embeddedness. Second, we derive an optimal enforcement policy against crime that exploits the geometric intricacies of the network structure connecting agents. For this purpose, we build on

[1] For example, Ken Burdett *et al.* (2003) and Chien-Chieh Huang *et al.* (2004) study the interaction between crime and unemployment, while Thierry Verdier and Yves Zenou (2004) analyze the impact of the land market on criminal activities. Others have focused on the education market (Lance Lochner, 2003) or on political economy aspects of crime (Ayse İmrohoroğlu *et al.*, 2004). Most of these models generate multiple equilibria that can explain why identical areas may end up with different amounts of crime.

the Beckerian incentives approach to crime behavior but let the cost to commit criminal offenses to be determined, in part, by one's network of criminal mates.[2] We then describe equilibrium behavior by resorting to standard network centrality measures in sociology. Finally, we introduce a new centrality measure that characterizes geometrically the optimal network-based enforcement policy. Our paper thus wedges a bridge between the economics of crime and the sociology literature on social networks.

The sociology literature on social networks is well-established and extremely active (see, in particular, Stanley Wasserman and Katherine Faust, 1994). One of the focus of this literature is to propose different measures of network centralities and to assert the descriptive and/or prescriptive suitability of each of these measures to each particular situation.[3] While these measures are mainly geometric in nature, our paper provides a behavioral foundation to the famous Bonacich's centrality measure[4] that we derive from a non-cooperative game in crime efforts on the network. More precisely, we show that the Nash equilibrium efforts of the crime-network game are proportional to the individual Bonacich centrality indexes, and we refer to it from now on as the *equilibrium Bonacich-centrality measure*.

In network games, the payoff interdependence is, at least in part, rooted in the network links across players (see, in particular, the recent literature surveys by Sanjeev Goyal, 2004 and Matthew O. Jackson, 2004). In general, at the Nash equilibria of a game, players' strategies subsume the payoff interdependence in a consistent manner. In the particular case of network games, equilibrium strategies should thus naturally reflect the players' network embededdness. For the crime network game we analyze, this relationship between equilibrium strategic behavior and network topology is straightforward and captured by the equilibrium Bonacich-centrality measure. This measure is an index of *connectivity* that not only takes into account the number of direct links a given criminal has but also all his indirect connections.[5] In our crime game, the network payoff interdependence is restricted to direct network mates. But, because clusters of direct friends overlap, this local payoff interdependence spreads all over the network. At equilibrium, individual decisions emanate from all the existing network chains of direct and indirect contacts stemming from each player, a feature characteristic of Bonacich centrality.

Because network chains of contacts often overlap, the values of individual centrality indexes are interrelated, which further translates into the interdependence of individual crime outcomes, and between individual and group (average) outcomes. This dependence of individual on group behavior is usually referred to as *peer effects* in the literature.[6] Peer effects are an intragroup

[2] See Antoni Calvó-Armengol and Yves Zenou (2004) for a first model with these features.

[3] See Steve P. Borgatti (2003) for a discussion on the lack of a systematic criterium to pick up the "right" network centrality measure for each particular situation.

[4] which has been proposed for nearly two decades ago in sociology by Phillip Bonacich (1987).

[5] There are, of course, other measures of centrality (for example the class of *betweenness* measures; see Wasserman and Faust, 2000).

[6] The empirical evidence collected so far suggests that peer effects are, indeed, quite strong in criminal decisions.

externality, *homogeneous* across group members, that captures the *average* influence that members exert on each other. In our model, though, the peer effect influence varies across criminals with their equilibrium-Bonacich centrality measure. The intragroup externality we obtain is *heterogeneous* across criminals, and this heterogeneity reflects asymmetries in network locations across group members.

The standard policy tool to reduce aggregate crime relies on the deterrence effects of punishment. By uniformly hardening the punishment costs borne by all criminals, the distribution of crime efforts shifts to the left and the average (and aggregate) crime level decreases. This homogeneous policy tackles average behavior explicitly and does not discriminate among criminals depending on their relative contribution to the aggregate crime level. Our previous results, though, associate a distribution of crime efforts to the network connecting them. In particular, the variance of crime efforts reflects the variance of network centralities. In this case, a targeted policy that discriminates among criminals depending on their relative network location, and removes a few suitably selected targets from this network, alters the whole distribution of crime efforts, not just shifting it. In many cases, it may yield to a sharper reduction in aggregate crime than standard deterrence efforts. In practice, the planner may want to identify optimal network targets to concentrate (scarce) investigatory resources on some particular individuals, or to isolate them from the rest of the group, either through leniency programs, social assistance programs, or incarceration.

To characterize the network optimal targets, we propose a new measure of network centrality, the *optimal inter-centrality measure*, that does not exist in the social network literature. This measure solves the planner's problem that consists in finding and getting rid of the *key player*, i.e., the criminal who, once removed, leads to the highest aggregate crime reduction. We show that the key player is, precisely, the individual with the highest optimal inter-centrality in the network.

Contrary to the equilibrium-Bonacich centrality index, this new centrality measure does not derive from strategic (individual) considerations, but from the planner's optimality (collective) concerns. The equilibrium Bonacich-centrality measure fails to internalize all the network payoff externalities criminals exert on each other, while the optimal inter-centrality measure internalizes them all. The optimal inter-centrality measure accounts not only for individual Bonacich centralities but also for cross-contributions to these equilibrium centralities. As such, the ranking of criminals according to their individual optimal inter-centrality measures, relevant for the selection of the optimal network target, need not always coincide with the ranking induced by individual equilibrium-Bonacich centralities. In other words, the key player is not necessarily the most active criminal. Indeed, removing a criminal from a network has both a direct and an indirect effect. First, less criminals contribute to the aggregate crime level. This is the direct effect. Second, the network topology is modified, and the remaining criminals adopt different crime efforts. This is the

See, for instance, Anne Case and Larry Katz (1991), Jens Ludwig *et al.* (2001) and Patrick Bayer *et al.* (2003). Building on the binary choice model of William Brock and Steve Durlauf (2001), Sibel Sirakaya (2004) identifies social interactions as the primary source of recidivist behavior in the United States indirect effect. The key player is the one with the highest overall effect.

At this point, it is important to note that, to implement the key-player policy, one does not need to have all the information about the exact structure of the network. Indeed, the planner does not need to know all the links each individual has but only needs to be able to rank criminals according to their inter-centrality measure.[7] This is less demanding in terms of information and it implies, in particular, that two different networks can lead to the same policy implication, i.e., the same key player to remove. Take for example a star-shaped network. Then it does not matter how many links has the central criminal, or whether some peripheral criminals have some direct link with each other, or even how large the network is. In all these cases, the planner will remove the central criminal because this is the key player −the criminal with the highest optimal inter-centrality measure. This is obviously an extreme case and in other networks one may need more information to identify the key player. But this simple example highlights the advantages of implementing a key-player policy.

We extend our characterization of optimal single player network removal for crime reduction crime, the *key player*, to optimal group removal, the *key group*. For this purpose, we generalize the optimal inter-centrality measure to groups of players. For a given group size, the key group is precisely the one with the highest value for such centrality measure among groups of exactly this size. Given that the individual optimal inter-centrality captures both direct and indirect effects on equilibrium Bonacich-centrality measures, the generalization to a group of the optimal inter-centrality measure needs to account (once and only once) for all the cross-contributions that arise both within and outside the group. For this reason, and contrarily to most centrality measures found in the literature, the group centrality index is not a straightforward aggregation of its members centrality indexes.

Because the geometric intricacies of the crime network are explicitly taken into account in the characterization of optimal network targets, the implications of our policy prescriptions are quite different from the standard deterrence-based policies, where both the apprehension probability and punishment are increased uniformly. We show that the key player (group) policy displays amplifying effects, and the gains following the judicious choice of the key player (group) go beyond the differences in optimal inter-centrality measures between the selected targets and any other criminals in the network. We also show that the relative gains from targeting the key player (group) instead of operating a selection at random of a criminal in the crime network increase with the variability in optimal inter-centrality measures across criminals. In other words, the

[7] Note that an undirected unweighted network is fully characterized by $n(n-1)/2$ values −the list of actual network links. We show that two $n-$ dimensional vectors aggregate this information in an enough informative manner for our purposes: first, to identify crime behavior − equilibrium-Bonacich centrality− and second, to identify optimal policy targets −optimal inter-centality. We further show that the only valuable information to identify the optimal target provided by the vector of optimal-inter-centralities is of *ordinal* nature, which further reduces the informational requirements on the network structure to effectively implement this policy.

key player (group) prescription is particularly well-suited for networks that display stark location asymmetries across nodes. Also, our policy prescriptions rely on centrality measures particularly robust to mispecifications in network data, and thus open the door to relatively accurate estimations of these measures with small samples of network data.

In the last part of the paper, we endogeneize the network connecting criminals by allowing players to join the labor market instead of committing criminal offenses. The model is now richer since, apart from punishment, the outside wage is an additional crime-reducing policy tool available to the planner. We show that the key player policy prescription now depends both on network features and on the wage level.

2 Crime network outcomes

2.1 The crime network game

The network There are n criminals. Some criminals know each other, and some do not. The collection of interpersonal relationships among criminals constitutes a crime network g. When i and j are directly connected, we set $g_{ij} = g_{ji} = 1$. When there is no direct connection between them, then $g_{ij} = g_{ji} = 0$. By convention, $g_{ii} = 0$.

The crime decision game Consider some crime network g. Criminals in the network decide how much effort to exert. We denote by e_i the crime effort level of criminal i, and by $\mathbf{e} = (e_1, ..., e_n)$ the population crime profile.

Following Becker (1968), we assume that criminals trade off the costs and benefits of criminal activities to take their crime effort decision. The expected crime gains to criminal i are given by:

$$u_i(\mathbf{e},g) = \underbrace{y_i(\mathbf{e})}_{proceeds} - \underbrace{p_i(\mathbf{e}, g)}_{apprehension} \underbrace{f}_{fine} \tag{1}$$

The individual proceeds $y_i(\mathbf{e})$ correspond to the gross crime payoffs of criminal i. Individual i gross payoff positively depends on i's crime involvement e_i, and on the whole population crime effort $\mathbf{e}$. The sign of the *global*[8] payoff interdependence may reflect either complementarities or substitutabilities in individual efforts. Substitutabilities may arise, for instance, in the case of property crime where individual criminals compete against each other for the same victims and booty. Complementarities are to be expected in conspiracy or terrorist activities, where individual criminals are part of a network organization pursuing a common goal.

The cost of committing crime $p_i(\mathbf{e}, g)f$ is also positively related to e_i as the apprehension probability increases with one's involvement in crime, hitherto, with one's exposure to deterrence. Moreover, and consistent with standard criminology theories (see e.g. Sutherland, 1947, Sarnecki,

[8] That is, across all criminals in the network.

2001, Warr, 2002), we assume that criminals improve illegal practice while interacting with their direct criminal mates. In words, $p_i(\mathbf{e}, g)$ reflects *local* complementarities in crime efforts across criminals directly connected through g.[9]

For sake of tractability, we restrict to the following simple expressions. More precisely, we set:

$$\begin{cases} y_i\left(\mathbf{e}\right) = e_i \left[1 - \gamma \sum_{j=1}^{n} e_j\right] \\ p_i(\mathbf{e},g) = p_0 e_i \left[1 - \lambda \sum_{j=1}^{n} g_{ij} e_j\right] \end{cases} \tag{2}$$

With these expression, we have:

$$\frac{\partial^2 y_i}{\partial e_i \partial e_j} = -\gamma \qquad \text{and} \qquad -\frac{\partial^2 p_i f}{\partial e_i \partial e_j} = \pi \lambda g_{ij},$$

where $\pi = p_0 f$ is the the marginal expected punishment cost for an isolated criminal.

The parameter $-\gamma < 0$ measures the intensity of the *global* interdependence on gross crime payoffs. Here, individual crime efforts are global strategic substitutes. The optimal crime effort of a given criminal thus decreases with the crime involvement of any other criminal in the network. The expression $\pi\lambda > 0$ captures the *local* strategic complementarity of efforts on the apprehension probability. This expression is non-zero only when $g_{ij} = 1$, that is, when criminals i and j are directly linked to each other.

Criminals choose their crime effort in $[0, \overline{e}]$, where $n\overline{e}\max\{\lambda, \gamma\} = 1$. This last technical condition on the parameters guarantees that $y_i\left(\mathbf{e}\right)$ and $p_i(\mathbf{e},g)$ are well-defined quantities.

2.2 Equilibrium and network centrality

The network adjacency matrix To any network g we can associate its adjacency matrix, that we denote by $\mathbf{G}$. The adjacency matrix is simply a matrix representation of a network. The coefficients of the matrix $\mathbf{G}$ are the g_{ij}s, $1 \leq i, j \leq n$. By definition, each cell in $\mathbf{G}$ takes on values zero or one, and the cell with coordinates (i, j) is equal to one if and only if i and j are directly linked in g, that is, $g_{ij} = 1$. Given our convention that $g_{ii} = 0$, the diagonal of $\mathbf{G}$ consists on zeros. Since $g_{ij} = g_{ji}$, the matrix $\mathbf{G}$ is symmetric.

The matrix $\mathbf{G}$ keeps track of the direct connections on the network g. Denote by $\mathbf{G}^k = \mathbf{G}^{(k \overset{times}{\cdots})}\mathbf{G}$ the kth power of the adjacency matrix $\mathbf{G}$, where k is some non-zero integer, and let $g_{ij}^{[k]}$ be the (i, j) cell of this matrix. The matrix $\mathbf{G}^k$ keeps track of the indirect connections on the network g. We say that there is an indirect connection, also denominated a *path*, between i and j

[9] See also William Brock and Stephen Durlauf (2001) for a global/local dichotomy in capturing social interactions and Yannis Ioannides (2002) for an exhaustive analysis of the effects of network topology in the Brock and Durlauf setting. Observe that all our results remain unchanged if the local network externalities enter the benefit function instead of the cost function in (2) as long as network payoffs reflect net strategic substituability. See our discussion in section 5.

on the network g if there exists a sequence of direct links on g connecting i to j.[10] The number of links in the sequence determines the length of the path between i and j. The coefficient $g_{ij}^{[k]}$ in the (i,j) cell of $\mathbf{G}^k$ gives the number of paths of length k on the network g between i and j. Note that, by definition, a path between i and j needs not follow the shortest possible route between those agents. For instance, suppose that i and j are directly linked in g. Then, the sequence of direct links $ij \to ji \to ij$ constitutes a path of length three between i and j.

Characterization of the equilibrium Denote by $\mathbf{I}$ the $n-$identity matrix, by $\mathbf{1}$ the $n-$dimensional column vector of ones, and by $\mathbf{1}^T$ its transpose. Then, $\mathbf{J} = \mathbf{1} \cdot \mathbf{1}^T$ is the $n-$dimensional matrix of ones.

Define $\phi = \pi\lambda/\gamma$. This ratio measures the relative strength of the local strategic complementarity of efforts with respect to the global strategic substitutability on criminals' payoffs.[11]

Proposition 1 *The interior Nash equilibria in pure strategies of the crime network game* $\mathbf{e}^*$ *are the solutions to the following system of n linear equations with n unknowns in matrix form:*

$$[\mathbf{J} + \mathbf{I} - \phi\mathbf{G}] \cdot \mathbf{e} = \frac{1-\pi}{\gamma}\mathbf{1} \tag{3}$$

There exists a unique $0 < \overline{\phi} \leq 1$ *and a finite set* $\mathcal{Z} \in I\!R$ *such that, for all* $0 \leq \phi < \overline{\phi}$ *and* $\phi \notin \mathcal{Z}$, *the set of interior Nash equilibria in pure strategies of the crime network game exists and is unique.*

The previous result characterizes the Nash equilibria in pure strategies of the crime network game, and provides conditions for their existence and uniqueness. Recall that network payoffs reflect substitutability at the global level with intensity γ, and complementarity at the local level with intensity $\pi\lambda$. When $\phi = \pi\lambda/\gamma \leq 1$, network payoffs thus reflect *net strategic substitutability*, that is, $\partial^2 u_i/\partial e_i \partial e_j \leq 0$, for all pair of players i and j, with a strict inequality when $\phi < 1$. Yet, the intensity of this effect varies across pairs of players i and j, from a lowest value equal to $-\gamma$ for indirectly connected players such that $g_{ij} = 0$, to a highest value equal to $-\gamma(1-\phi)$ for directly connected players such that $g_{ij} = 1$. The interior Nash equilibrium reflects these differences in the relative intensities of the network payoff strategic substitutability across different pairs of players, and the balance of these differences as a function of the network geometry.

At equilibrium, the marginal gross crime gains equal the marginal punishment cost for each criminal, that is,

$$1 - \gamma\sum_{j=1}^{n} e_j - \gamma e_i = \pi\left[1 - \lambda\sum_{j=1}^{n} g_{ij}e_j\right], \text{ for all } i = 1, ..., n.$$

[10] Formally, a path between i and j is a sequence $i_0, ..., i_k$, $k \geq 1$ such that $i_0 = i$, $i_k = j$, and $g_{i_p i_{p+1}} = 1$, for all $0 \leq p \leq k-1$.

[11] All proofs of propositions and lemmata are given in the Appendix.

This is equivalent to the following vectorial equality:

$$\mathbf{1}-\gamma\left[\mathbf{J}+\mathbf{I}\right]\cdot\mathbf{e}=\pi\mathbf{1}-\pi\lambda\mathbf{G}\cdot\mathbf{e},$$

which, in turn, corresponds to (3). The matrix $\mathbf{J}$ captures the *global* payoff interdependencies, while the matrix $\mathbf{G}$ stands for the *local* network synergies.

The condition $\phi\notin\mathcal{Z}$ guarantees that the system (3) has a unique solution. We refer to the situations in which $\phi\notin\mathcal{Z}$ as generic situations. Since $\mathcal{Z}$ is a finite set, the whole set of nongeneric situations has Lebesgue measure of zero. From now on, we restrict to generic situations.

Comparative statics In Proposition 1, the individual and aggregate crime levels depend on the underlying network g connecting them through the adjacency matrix $\mathbf{G}$ in (3). The next result establishes a positive relationship between the equilibrium aggregate crime level and the network pattern of connections.

Proposition 2 *Let g and g' such that $g\subset g'$. At equilibrium, the total crime level under g' is strictly higher than that under g.*

Consider two nested networks g and g' such that $g\subset g'$. Then, either g and g' connect the same number of criminals but there are more direct links between them in g' than in g, or g' brings additional individuals into the pool of criminals already connected by g, or both. Proposition 2 shows that the density of network links and the network size (or boundaries) affect positively aggregate crime, a feature often referred to as the *social multiplier effect*.[12]

The intuition for this result is as follows. Consider two nested networks g and g', where $g\subset g'$. Then, any pair of players not directly connected in g' is not directly connected in g, and the total number of pairs of directly connected players in g' is higher than that in g. We know that crime efforts are strategic substitutes for any pair of players i and j, and that the strength of this relationship is lower when players are directly connected with each other.[13] Indeed, direct connections are the source of local complementarities that counter, at least in part, the global payoff substitutability. When $g\subset g'$ then, necessarily, the payoff cross-derivative between i and j is higher or equal in g' than in g for all players i and j, and it is strictly higher for some pairs of players (those directly connected in g' but not in g). As a result, players can rip more local complementarities in g' than in g, and equilibrium aggregate crime is higher in g' than in g.

[12] See, for instance, Edward Glaeser, Bruce Sacerdote and José Scheinkman (2003), and references therein.

[13] Recall from the discussion above that:

$$\frac{\partial^2 u_i(\mathbf{e})}{\partial e_i\partial e_j}=\begin{cases}-\gamma,\text{ when } g_{ij}=0\\ -\gamma(1-\phi),\text{ when } g_{ij}=1\end{cases}$$

where $0\leq-\gamma\leq-\gamma(1-\phi)\leq 0$, when $0\leq\phi\leq 1$.

A network centrality measure In a game setting, payoffs are interdependent, and the individual equilibrium strategies adopted at any equilibrium of such game subsume this interdependence in a consistent manner.

In the crime network game, the payoff interdependence is, in part, rooted in the network links through which know-how is being shared. We should thus expect the individual equilibrium crime levels to reflect the criminals' network embededdness. To clarify this relationship between network location and equilibrium outcomes,[14] we first define a useful network centrality measure.

Let $\mathbf{G}$ be the adjacency matrix of a crime network g. Recall that the coefficients of $\mathbf{G}^k$ give the number of paths of length k in g between any two pair of criminals connected by g. For all $k \geq 0$, define:

$$\boldsymbol{\beta}^k(g) = \mathbf{G}^k \cdot \mathbf{1}.$$

By definition, the ith coordinate of $\boldsymbol{\beta}^k(g)$ is equal to $\beta_i^k(g) = \sum_{j=1}^n g_{ij}^{[k]}$, and counts the number of direct and indirect paths of length k in g starting from i.

For sufficiently low values of ϕ, we can define the following vector:

$$\boldsymbol{\beta}(g, \phi) = \sum_{k=0}^{+\infty} \phi^k \boldsymbol{\beta}^k(g) = \sum_{k=0}^{+\infty} \phi^k \mathbf{G}^k \cdot \mathbf{1} = [\mathbf{I} - \phi \mathbf{G}]^{-1} \cdot \mathbf{1}, \tag{4}$$

Now, the ith coordinate of $\boldsymbol{\beta}(g, \phi)$ is equal to $\beta_i(g, \phi) = \sum_{k=0}^{+\infty} \phi^k \beta_i^k(g)$, and counts the *total* number of direct and indirect paths in g starting from i for all possible paths lengths. In this expression, the paths of length k are weighted by the geometrically decreasing factor ϕ^k.

The vector $\boldsymbol{\beta}(g, \phi)$ is a variation of the network centrality measure due to Philipp Bonacich (1987).[15] Because it is derived from a Nash equilibrium, it is referred to as the *equilibrium Bonacich-centrality measure*. Over the past years, social network theorists have proposed a number of centrality measures to account for the variability across network locations.[16] Roughly, these indices encompass two dimensions of network centrality: connectivity and betweenness. The simplest index of connectivity is the number of direct links stemming from each node in a network, while betweenness centrality keeps track of the number of optimal paths across (or from) every node.[17]

[14] Recall that the network geometry is explicitly present in the system of equations (3) that characterize the equilibria of the crime network game through the adjacency matrix $\mathbf{G}$ of the crime network g.

[15] In fact, $\boldsymbol{\beta}(g, \phi)$ is obtained from Bonacich's measure by an affine transformation. More precisely, Bonacich defines the following network centrality measure:

$$\boldsymbol{\psi}(g, a, b) = a\,[\mathbf{I} - b\mathbf{G}]^{-1}\,\mathbf{G} \cdot \mathbf{1}.$$

Therefore, $\boldsymbol{\beta}(g, \phi) = \mathbf{1} + \phi \boldsymbol{\psi}(g, 1, \phi) = \mathbf{1} + \boldsymbol{\kappa}(g, \phi)$, where $\boldsymbol{\kappa}(g, \phi)$ is an early measure of network status introduced by Leo Katz (1953). See also Roger Guimerà *et al.* (2001) and Mark Newman (2003) for related network centrality measures.

[16] See Wasserman and Faust (1994) and references therein.

[17] See Linton Freeman (1979) for an example of betweenness centrality, equal to the mean of the shortest-path distance between some given node and all the other nodes that can be reached in the network. Different concepts of path optimality (shortest-path, maximal-flow, etc.) lead to different betweenness measures.

Our centrality measure $\boldsymbol{\beta}(g,\phi)$ is an index of connectivity (and not betweenness). It counts the number of *any* path stemming from a given node, not just optimal paths. These numbers are then weighted by a factor ϕ^k that decays geometrically with the path length. If ϕ is low enough, the infinite sum in (4) takes on a finite value. For very small values of ϕ, the coordinates of $\boldsymbol{\beta}(g,\phi)$ are an affine function of the number of direct links of every criminal in the network. The higher the value of ϕ, the higher the contribution of indirect and distant links to the centrality measure of any criminal in the network.

From network structure to crime outcomes The following result establishes that the equilibrium individual effort levels of the crime network game are proportional to the equilibrium Bonacich-centrality measure of each criminal. It thus relates strategic equilibrium behavior to network topology.

Define $\beta(g,\phi) = \mathbf{1}^T \cdot \boldsymbol{\beta}(g,\phi)$. This is the sum of the centrality measures for all criminals connected through g.

Proposition 3 *There exists a unique* $0 < \widehat{\phi} \leq 1$ *such that, for all* $0 \leq \phi < \widehat{\phi}$*, the unique Nash equilibrium strategies of the crime network game are given by:*

$$e_i^* = \frac{1-\pi}{\gamma} \frac{\beta_i(g,\phi)}{1+\beta(g,\phi)}, \text{ for all } i = 1,...,n. \tag{5}$$

The equilibrium Bonacich-centrality measure $\boldsymbol{\beta}(g,\phi)$ is thus the relevant network characteristic that shapes equilibrium behavior. This measure of centrality reflects both the direct and the indirect network links stemming from each criminal. In (1), though, the local payoff interdependence is restricted to direct network mates, and equilibrium behavior should only integrate this local interdependencies. Yet, because clusters of direct friends overlap, the local payoff interdependence spreads all over the network.[18] As a result, at equilibrium, individual decisions emanate from all the existing network chains of direct and indirect contacts. Because individual decisions feed into each other along any network path, every such path (not only shortest-paths, for instance) shapes the equilibrium behavior of criminals.

The condition $\phi < \widehat{\phi}$ relates the payoff function to the network topology. This is not surprising, as this condition allows us to characterize Nash outcomes, which depend on the payoff structure, in terms of centrality measures, which reflect the network topology. When this condition holds, the ratio of the local to the global payoff interdependence $\phi = \pi\lambda/\gamma$ is lower than the inverse of the highest eigenvalue of the adjacency matrix $\mathbf{G}$ of the network g. In this case (and only then), the matrix $[\mathbf{I}-\phi\mathbf{G}]^{-1}$ can be developed into the infinite sum $\sum_{k\geq 0}\phi^k\mathbf{G}^k$, which brings the Bonacich-centrality measure into the picture. We provide in the appendix an analytical expression for the

[18] At equilibrium, i's effort decision depends on j's effort decision, for all j such that $g_{ij} = 1$. But j's effort decision depends, in turn, on k's effort decision, for all k such that $g_{jk} = 1$. Therefore, i's decision depends (indirectly) on k's decision, for all k such that $g_{ik}^{[2]} = 1$. And so on.

upper bound $\widehat{\phi}$ as a function only of the total number of direct links in g and the number of direct links of the least connected node in g. In particular, it is readily checked that $\widehat{\phi} \geq 1/k$ for the whole class of regular networks where all players have exactly k direct links, independent of the size n of the population connected by these networks.

Observe that, when criminals hold different location in the network, they will exert different crime efforts. Equation (5) then implies that the ranking of equilibrium crime efforts across criminals reflects exactly the ranking of their network centrality measures. Network structure is thus a determinant of criminal outcomes.

Denote by $e^* = \mathbf{1}^T \cdot \mathbf{e}^*$ the equilibrium aggregate crime level. Together with (5), we get:

$$e_i^* = \frac{\beta_i(g, \phi)}{\beta(g, \phi)} e^*.$$

In words, the individual contribution of each player to aggregate crime is proportional to his network centrality measure. The dependence of individual outcomes on group behavior is usually referred to as *peer effects*.[19] In a standard peer effect model, though, the dependence of individual to group outcomes is the same for all individuals. The intragroup externality is *homogeneous* across group members, and corresponds to a group *average* influence that members exert on each other. Here, the strength of the peer effect influence varies across criminals according to their location in the network, where the relevant index for network position is (a variation of) the Bonacich centrality measure. More central players have higher exposure to the rest of the group and experience a higher involvement in crime, and vice-versa. The intragroup externality is thus *heterogeneous* across criminals, and this heterogeneity reflects asymmetries in network locations across group members. Network centrality indicates how peer effects are distributed within the group and thus captures the *variance* of this intragroup externality.

Consistent with the predictions of our model, a recent empirical study by Haynie (2001) shows that structural properties of friendship networks indeed condition the association between friends' delinquency and an individual's own delinquent behavior. More precisely, she finds that Bonacich centrality, net of other individual effects, accounts for 21 percent of the observed differences in adolescents delinquency-peer association.[20] Also, by analyzing the network organization of conspiracy, Wayne Baker and Robert Faulkner (1993) show that a measure of network centrality based on direct links predicts the individual probability to be apprehended and convicted, and the magnitude of the fine.

[19] The empirical evidence collected so far suggests that peer effects are, indeed, very strong in criminal decisions. See, for instance, Anne Case and Larry Katz (1991), Jens Ludwig *et al.* (2001) and Patrick Bayer *et al.* (2003).

[20] Data from the National Longitudinal Study of Adolescent Health, United States, 1994-1995.

Example To illustrate the previous results, consider the following crime network g with three criminals, where agent 1 holds a central position whereas agents 2 and 3 are peripherals.

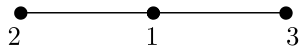


Figure 1

The adjacency matrix for this crime network is the following:

$$\mathbf{G} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

Its is a straightforward algebra exercise to compute the powers of this matrix, which are:

$$\mathbf{G}^{2k} = \begin{bmatrix} 2^k & 0 & 0 \\ 0 & 2^{k-1} & 2^{k-1} \\ 0 & 2^{k-1} & 2^{k-1} \end{bmatrix} \quad \text{and} \quad \mathbf{G}^{2k+1} = \begin{bmatrix} 0 & 2^k & 2^k \\ 2^k & 0 & 0 \\ 2^k & 0 & 0 \end{bmatrix}, k \geq 1.$$

For instance, we deduce from $\mathbf{G}^3$ that there are exactly two paths of length three between criminals 1 and 2, namely, $12 \to 21 \to 12$ and $12 \to 23 \to 32$. Obviously, there is no path of this length (and, in general, of odd length) from any criminal to himself. We can now compute the criminals' centrality measures using (4). We obtain:[21]

$$\begin{aligned} \beta_1(g, \phi) &= \sum_{k=0}^{+\infty} \left[\phi^{2k} 2^k + \phi^{2k+1} 2^{k+1} \right] = \frac{1 + 2\phi}{1 - 2\phi^2} \\ \beta_2(g, \phi) &= \beta_3(g, \phi) = \sum_{k=0}^{+\infty} \left[\phi^{2k} 2^k + \phi^{2k+1} 2^k \right] = \frac{1 + \phi}{1 - 2\phi^2} \end{aligned}$$

According to intuition, criminal 1 has the highest centrality measure. All centrality measures β_is increase with ϕ, and so does the ratio β_1/β_2 of agent 1's centrality with respect to any other criminal, as the contribution of indirect paths to centrality increases with ϕ. Then, using expressions (5) in Proposition 3, we obtain the following crime efforts at equilibrium:

$$e_1^* = \frac{1 - \pi}{4\gamma} \frac{1 + 2\phi}{1 + \phi} \quad \text{and} \quad e_2^* = e_3^* = \frac{1 - \pi}{4\gamma}.$$

As expected, the crime effort exerted by criminal 1, the most central player, is the highest one.

[21] Note that this centrality measures are only well-defined when $\phi < 1/\sqrt{2}$. This upper bound, which guarantees that the infinite sums converge, can be obtained by inspection and, more generally, can be deduced from the general expression in the proof of Proposition 3.

3 Crime network policies

3.1 Finding the key player

A network-based policy The standard policy tool to reduce aggregate crime relies on the deterrence effects of punishment (see for example Becker, 1968). Formally, an increase in π, which translates into an increase in ϕ, amounts to hardening punishment costs borne by criminals. Our previous results associate a distribution of crime efforts across criminals to any crime network connecting them. In this case, an increase in ϕ affects all criminal decisions simultaneously and shifts the whole crime efforts distribution to the left, thus reducing the average (and the aggregate) crime level.

In our model, though, crime behavior is tightly rooted in the network structure. When all criminals hold homogeneous positions in the crime network, they all exert a similar crime effort. In this case, the above-mentioned policy, that tackles average behavior and does not discriminate among criminals depending on their relative contribution to the aggregate crime level, may be appropriate. However, if criminals hold very heterogeneous positions in the crime network, they contribute very differently to the aggregate crime level. The variance of efforts is higher. In this case, we could expect a sharp reduction in average crime by directly removing a criminal from the network and thus altering the whole distribution of crime efforts, not just shifting it. A targeted policy that discriminates among criminals depending on their location in the network may then be more appropriate.[22]

In what follows, we first provide a simple geometric criterium to identify the optimal target, and then compare the new policy with the more standard one.

The planner's problem Denote by $e^*(g, \phi) = \mathbf{1}^T \cdot \mathbf{e}^*$ the equilibrium aggregate crime level corresponding to a network g. The planner's problem is to reduce the overall equilibrium crime level $e^*(g, \phi)$ with the policy tools available. Standard policy tools consist on increasing the deterrence effort ϕ.

Here, we examine an alternative policy that consists on manipulating the network g that connects criminals. We first consider the simple case where the planner can eliminate only one criminal i from the crime network. By eliminating criminal i, the network g changes its shape as all the direct links in g stemming from i also disappear. We denote by g^{-i} the resulting network, where $g_{jk}^{-i} = 1$ if and only if both $g_{jk} = 1$ and $j \neq i \neq k$. When i is eliminated, the resulting overall crime level is $e^*(g^{-i}, \phi)$.

[22] See Réka Albert *et al.* (2000) for a heuristical and numerical analysis of the relative network disruption effects of a coordinated attack versus random failures in large systems networks, such as the World Wide Web or the internet network, as a function of the underlying network topology. Béla Bollobás and Oliver Riordan (2003) provide a thorough mathematical account of these results.

The planner's objective is to generate the highest possible reduction in aggregate crime level by picking the appropriate criminal. Formally, the planner's problem is the following:

$$\max\{e^*(g,\phi) - e^*(g^{-i},\phi) \mid i = 1,...,n\},$$

which, when the original crime network g is fixed, is equivalent to:

$$\min\{e^*(g^{-i},\phi) \mid i = 1,...,n\} \tag{6}$$

This is a finite optimization problem, that admits at least one solution. Let i^* be a solution to (6). We call criminal i^* the *key player*. Removing criminal i^* from the initial crime network g, instead of picking any other criminal, has the highest overall impact on the aggregate crime level. Identifying the key player requires the comparison of the maximal aggregate equilibrium outcomes across n different crime network games, where the games differ in that a different criminal is removed each time from the initial network, each removal leading, in turn, to a different network setting.

In what follows, we provide a simple geometric characterization of the key player in the original crime network g.

A geometric characterization of the key player When criminal i is removed from g, the new crime network is g^{-i}. By (5), the aggregate crime level becomes:

$$e^*(g^{-i},\phi) = \frac{1-\pi}{\gamma}\frac{\beta(g^{-i},\phi)}{1+\beta(g^{-i},\phi)}$$

Given that the function $f(x) = x/(1+x)$ is increasing in x, the planner's problem (6) can be reformulated as:

$$\min\{\beta(g^{-i},\phi) \mid i = 1,...,n\} \tag{7}$$

Thanks to Proposition 3 that relates network structure to crime outcomes, the planner's original objective of reducing crime translates into a geometric problem of decreasing the network aggregate centrality measure $\beta(g^{-i},\phi)$.

When one criminal is eliminated from the current crime pool, the impact on the overall crime level is twofold. First, aggregate crime decreases as one criminal −the one being eliminated− does not contribute anymore to the group outcome. This is a direct effect. Second, with the removal of this criminal, the topology of the network connecting the remaining set of criminals is altered. As a result, the centrality measure accruing to each of them is modified, and their individual involvement in crime changes accordingly. This is an indirect effect.

The key player is the one inducing the highest aggregate crime reduction. Given that criminal removal has both a direct and an indirect on the group outcome, the choice of the key player results from a compromise between both effects. In particular, the key player need not necessarily be the one exerting the highest crime effort or, equivalently, the one with the highest centrality measure.

We now define a new network centrality measure $\boldsymbol{\theta}(g, \phi)$ that will happen to solve this compromise. This measure of centrality, that we refer to as *optimal inter-centrality measure*, reflects both one's centrality and one's contribution to the others' centrality.

For all $i \neq j$ and non-zero integer k define:

$$\beta_{ij}^{k}(g, \phi) = \sum_{p=1}^{k} g_{ij}^{[p]} \beta_{j}^{k-p}(g, \phi).$$

Recall that $\beta_j^{k-p}(g)$ is equal to the number of paths in g of length $k-p$ starting from j. Also, $g_{ij}^{[p]} \geq 1$ if and only if there exists at least one path in g of length p between i and j. Altogether, $g_{ij}^{[p]} \beta_j^{k-p}(g, \phi)$ is equal to the number of paths of length k that start from i and cross through j after p links. Therefore, $\beta_{ij}^k(g, \phi)$ is the total number of paths of length k that start at i and cross through j (at least once) or end at j. Summing across all weighted path lengths leads to the following expression:

$$\beta_{ij}(g, \phi) = \sum_{k=1}^{+\infty} \phi^k \beta_{ij}^k(g, \phi).$$

By definition, it is readily checked that:

$$\beta_{ij}(g, \phi) = \beta_i(g, \phi) - \beta_i(g^{-j}, \phi). \tag{8}$$

In words, the contribution of criminal j to criminal i's centrality in g is equal to the difference of criminal i's centrality in g and in g^{-j}, where criminal j has been removed.

Definition 1 *For all network g and for all i, let $\theta_i(g, \phi) = \underbrace{\beta_i(g, \phi)}_{direct} + \sum_{j \neq i} \underbrace{\beta_{ji}(g, \phi)}_{indirect}$.*

The inter-centrality measure $\theta_i(g)$ of criminal i is the sum of i's centrality measures in g, and i's contribution to the centrality measure of every other criminal $j \neq i$ also in g. It accounts both for one's exposure to the rest of the group and for one's contribution to every other exposure.

The following result establishes that inter-centrality captures, in an meaningful way, the two dimensions of the removal of a criminal from a network, namely, the direct effect on crime and the indirect effect on others' crime involvement.

Proposition 4 *A player i^* is the key player that solves (7) if and only if i^* is a criminal with the highest inter-centrality in g, that is, $\theta_{i^*}(g, \phi) \geq \theta_i(g, \phi)$, for all $i = 1, ..., n$.*[23]

The previous result provides a geometric characterization of the key player.

[23] Note that there may be more than one key player as different criminals may display the same value for their inter-centrality measure.

Example Consider the network g in Figure 2 with eleven criminals.

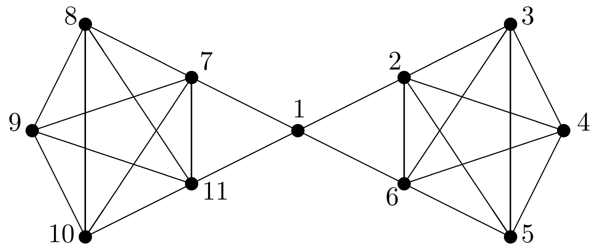


Figure 2

We distinguish three different types of equivalent actors in this network, which are the following:

Type	Players
1	1
2	2, 6, 7 and 11
3	3, 4, 5, 8, 9 and 10

From a macro-structural perspective, type-1 and type-3 criminals are identical: they all have four direct links, while type -2 criminals have five direct links each. From a micro-structural perspective, though, criminal 1 plays a critical role by bridging together two closed-knit (fully intraconnected) communities of five criminals each. By removing criminal 1, the network is maximally disrupted as these two communities become totally disconnected, while by removing any of the type-2 criminals, the resulting network has the lowest aggregate number of network links.

We identify the key player in this network of criminals. If the choice of the key player were solely governed by the *direct* effect of criminal removal on aggregate crime, type-2 criminals would be the natural candidates. Indeed, these are the ones with the highest number of direct connections. But the choice of the key player needs also to take into account the *indirect* effect on aggregate crime reduction induced by the network restructuring that follows the removal of one criminal from the original network. Because of his communities' bridging role, criminal 1 is also a possible candidate for the preferred policy target.

Table 1 computes, for criminals of types 1, 2 and 3 the value of crime efforts e_i^*, centrality measures $\beta_i(g, \phi)$ and inter-centrality measures $\theta_i(g, \phi)$ for different values of ϕ and with $\gamma = \lambda = 1$.

In each column, a star identifies the highest value.[24]

ϕ	0.1			0.2		
Player Type	e_i^*	β_i	θ_i	e_i^*	β_i	θ_i
1	0.077	1.75	2.92	0.072	8.33	41.67*
2	0.082*	1.88*	3.28*	0.079*	9.17*	40.33
3	0.075	1.72	2.79	0.067	7.78	32.67

First note that type-2 criminals always display the highest $\beta-$centrality measure. These criminals have the highest number of direct connections. Besides, they are directly connected to the bridge criminal 1, which gives them access to a very wide and diversified span of indirect connections. Altogether, they are the most $\beta-$central criminals.

For low values of ϕ, the direct effect on crime reduction prevails, and type-2 criminals are the key players $-$those with highest optimal inter-centrality measure θ. When ϕ is higher, though, the most active criminals are not anymore the key players. Now, indirect effects matter a lot, and eliminating criminal 1 has the highest joint direct and indirect effect on aggregate crime reduction.

When the punishment cost ϕ is low, criminals transfer their know-how only at a very local level, with their direct criminal mates. When ϕ increases, criminals counter the higher deterrence they face by spreading their know-how further away in the network and establishing synergies with criminals located in distant parts of the social setting. In this latter case, the optimal targeted policy is the one that maximally disrupts the crime network, thus harming the most its know-how transferring ability.

Note that the network g^{-1} has twenty different links, while g^{-2} has nineteen links. In fact, when ϕ is small enough, the key player problem minimizes the number of remaining links in a network, which explains why type-2 criminals are the key player when $\phi = 0.1$ in this example.

Formally, let $n_i(g) = \sum_j g_{ij}$. This is the number of direct contacts of criminal i in the network g. Let $n(g) = \frac{1}{2}\sum_i n_i(g)$. This is the total number of links in g. We denote by $o(x)$ a function that converges to zero when x tend to zero at a rate faster than x. Formally, $\lim_{x \to 0} o(x)/x = 0$.

Lemma 1 *When $\phi \to 0$, we have $\beta_i(g, \phi) = 1 + \phi n_i(g) + o(\phi)$. Then, the key player i^* solves $\min_{i \in N} n(g^{-i})$.*

Optimal-intercentrality versus equilibrium-Bonacich centrality The individual Nash equilibrium efforts of the crime-network game are proportional to the *equilibrium Bonacich-centrality* network measures, while the key player is the criminal with the highest *optimal inter-centrality measure*. As the previous example illustrates, these two measures need not coincide. This is not surprising, as both measures differ substantially in their foundation. Whereas the equilibrium-Bonacich

[24] From the proof of Proposition 3, we can compute the highest possible value for ϕ compatible with our definition of centrality measures, equal to $\widehat{\phi} = \frac{2}{3+\sqrt{41}} \simeq 0,213$.

centrality index derives from strategic individual considerations, the optimal inter-centrality measure solves the planner's optimality collective concerns. In particular, the equilibrium Bonacich-centrality measure fails to internalize all the network payoff externalities criminals exert on each other, while the optimal inter-centrality measure internalizes them all. More formally, the measure $\boldsymbol{\theta}(g, \phi)$ goes beyond the measure $\boldsymbol{\beta}(g, \phi)$ by keeping track of all the cross-contributions that arise between its coordinates $\beta_1(g, \phi), ..., \beta_n(g, \phi)$.

We now clarify the relationship between $\boldsymbol{\theta}(g, \phi)$ and $\boldsymbol{\beta}(g, \phi)$.

Define the following matrix:

$$\mathbf{M}(g, \phi) = [\mathbf{I} - \phi \mathbf{G}]^{-1} = \sum_{k=0}^{+\infty} \phi^k \mathbf{G}^k.$$

The elements of this matrix, denoted by $m_{ij}(g, \phi)$ and given by $m_{ij}(g, \phi) = \sum_{k=0}^{+\infty} \phi^k g_{ij}^{[k]}$ count the weighted number of paths in g starting at i and ending at j, for all $1 \leq i, j \leq n$.

The equilibrium-Bonacich centrality of player i counts the *total* number of direct and indirect paths in g starting from i for all possible paths lengths.[25] Recall from (4) that $\boldsymbol{\beta}(g, \phi) = \mathbf{M}(g, \phi) \cdot \mathbf{1}$. Therefore:

$$\beta_i(g, \phi) = \underbrace{m_{ii}(g, \phi)}_{self-loops} + \underbrace{\sum_{j \neq i} m_{ij}(g, \phi)}_{out-paths}.$$

In words, we can distinguish between two different types of paths that start from i in g. One one hand, *self-loops* that start from i and end up at the same starting point i; these paths account for a proportion $m_{ii}(g, \phi)/\beta_i(g, \phi)$ of the total Bonacich centrality of player i. On the other hand, *out-paths* that start from i but end up at some other node $j \neq i$; they account for a (complementary) share $1 - m_{ii}(g, \phi)/\beta_i(g, \phi)$ of i's Bonacich centrality.

Self-loops correspond to the n diagonal terms $(m_{11}(g, \phi), ..., m_{11}(g, \phi))$ of the matrix $\mathbf{M}(g, \phi)$, and each self-loop is associated to only one coordinate of the $n-$dimensional vector $\boldsymbol{\beta}(g, \phi)$ with coordinates $(\beta_1(g, \phi), ..., \beta_n(g, \phi))$. Out-paths correspond to the out-of-diagonal terms of $\mathbf{M}(g, \phi)$, collected row by row. Given the symmetry of the matrix $\mathbf{M}(g, \phi)$, the out-paths components of $\beta_i(g, \phi)$ and $\beta_j(g, \phi)$ share a common factor $m_{ij}(g, \phi) = m_{ji}(g, \phi)$, for all $i \neq j$. Out-paths thus reflect the cross-contributions in the individual equilibrium-Bonacich centralities and, as such, should also enter the calculation of individual optimal-intercentralities which, precisely, subsume all such cross-contributions.

The following result relates optimal-intercentrality, self-loops and equilibrium-Bonacich centrality.

Proposition 5 *For all network g we have $\theta_i(g, \phi) = \beta_i(g, \phi)^2 / m_{ii}(g, \phi)$, for all $i = 1, ..., n$.*

[25] Where paths of length k are weighted by the geometrically decreasing factor ϕ^k.

Holding $\beta_i(g, \phi)$ fixed, the intercentrality $\theta_i(g, \phi)$ of player i decreases with the proportion $m_{ii}(g, \phi)/\beta_i(g, \phi)$ of i's Bonacich centrality due to self-loops, and increases with the fraction of i's centrality amenable to out-paths.

3.2 Comparing policies

The cost of finding the key player Given a crime network g and a punishment cost ϕ, the ranking of criminals according to their individual inter-centrality measure $\theta_i(g, \phi)$s provides a criterium for the selection of an optimal target in the network. Implementing such a network-based policy has obviously its costs. Indeed, the computation of the inter-centrality measures relies on the knowledge of the adjacency matrix of the crime network. This matrix is obtained from sociometric data that identifies the network links between criminals. It is important to note that sociometric data on crime is available in many cases. For instance, Haynie (2001) uses friendship data to identify delinquent peer networks for adolescents in 134 schools in the U.S. that participated in an in-school survey in the 1990's. Sarnecki (2001) provides a comprehensive study of co-offending relations and corresponding network structure for football hooligans and right-wing extremists in Stockholm. Baker and Faulkner (1993) reconstruct the structure of conspiracy networks for three well-known cases of collusion in the heavy electrical equipment industry in the U.S. Finally, Valdis Krebs (2002) maps the network of terrorist cells behind the tragic event of September 11th, 2001. In all these cases, one may directly use the available data to compute the inter-centrality measures.[26]

In some other cases, though, *ad hoc* information gathering programs have to be implemented. Interestingly, Elizabeth Costebander and Thomas Valente (2003) show that centrality measures based on connectivity (rather than betweenness), such as $\boldsymbol{\beta}$ and $\boldsymbol{\theta}$, are robust to mispecifications in sociometric data, and thus open the door to estimations of centrality measures with (relatively small) samples of network data.[27] This, obviously, reduces the cost of identifying the key player.

Key player versus random target To fully assess the relevance of the key player crime policy, we also need to evaluate the relative returns from following this network targeted policy. For this purpose, we compare the reduction in aggregate crime following the elimination of the key player with respect to the expected consequences when the target is selected randomly.

For each criminal i in the crime network, define:

$$\eta_i(g, \phi) = n \frac{e^*(g, \phi) - e^*(g^{-i}, \phi)}{\sum_{j=1}^{n} \left[e^*(g, \phi) - e^*(g^{-j}, \phi)\right]}.$$

This is the ratio of returns (in crime reduction) when i is the selected target versus a random selection with uniform probability for all criminals in the network.

[26] In fact, Haynie (2001) conducts regressions where the Bonacich centrality measure is taken as an explanatory variable for delinquent crime decisions.

[27] See, also, Tami Carpenter, George Karakostas and David Shallcross (2002) for a discussion on algorithms that deals with data uncertainty in terrorist networks.

Denote by $\overline{\theta}(g,\phi)$ the average of the inter-centrality measures in network g, and by $\sigma_\theta(g,\phi)$ the standard deviation of the distribution of this inter-centrality measures. The following result establishes a lower bound on the ratio of returns in crime reduction when the key player is removed.

Proposition 6 *Let i^* be the key player in g for a given ϕ. Then,*

$$\eta_{i^*}(g,\phi) \geq 1 + \frac{\sigma_\theta(g,\phi)}{\overline{\theta}(g,\theta)}.$$

The relative gains from targeting the key player instead of operating a selection at random in the crime network increase with the variability in inter-centrality measures across criminals as captured by $\sigma_\theta(g,\phi)$. In other words, the key player prescription is particularly well-suited for networks that display stark location asymmetries across nodes. In these cases, it is more likely than the relative gains from implementing such a policy compensate for its relative costs.

Key player versus standard deterrence policy Of course, the planner can also reduce aggregate crime by implementing a standard deterrence policy, that is, increasing punishment costs ϕ. The impact on aggregate crime following an increase in ϕ, though, results from the combination of two effects that work in opposite directions as can be seen in (5). First, the individual probability to be apprehended, and thus the punishment costs borne by each criminal, increase with ϕ. This is a *direct negative* effect. Second, when ϕ increases, criminals react strategically by acquiring a better crime technology to thwart the higher deterrence they now face. The improvement in crime technology stems from more intense know-how inflows and transfers in the crime network. Each criminal centrality measure $\beta_i(g,\phi)$ increases, which translates into a higher crime involvement for each criminal. This is an *indirect positive* effect on aggregate crime that mitigates the direct negative effect.

On the contrary, the key player removal policy has a straightforward effect on crime reduction, with no countervailing effect. Indeed, when a criminal is removed from the network, the inter-centrality measures of all the criminals that remain active are reduced, that is, $\theta_j(g^{-i^*},\phi) \leq \theta_j(g,\phi)$, for all $j \neq i^*$, which triggers a decrease in crime involvement for all of them. Moreover, when criminal i^* is removed from the crime network, the corresponding ratio of aggregate crime reduction with respect to the network centrality reduction is an increasing function of the inter-centrality measure $\theta_i(g,\phi)$ of this criminal. Formally,

$$\frac{\partial}{\partial \theta_i(g,\phi)} \left[\frac{e^*(g,\phi) - e^*(g^{-i},\phi)}{\beta(g,\phi) - \beta(g^{-i},\phi)} \right] > 0.$$

In words, the target policy displays amplifying effects, and the gains following the judicious choice of the key player (the one with highest inter-centrality measure) go beyond the differences in inter-centrality measures between this player and any other criminal in the network.

3.3 From individual key player to key group

So far, we have characterized *optimal single player removal* from the network to reduce crime, a *key player*. We now characterize *optimal group removal* from the network, a *key group*.

The general planner's problem Given a group size $1 \leq s \leq n-1$, the planner's objective is to generate the highest possible reduction in aggregate crime level by picking a subset of criminals of exactly this size.

Let $N = \{1, ..., n\}$. Formally, the planner's problem is the following:

$$\max\{e^*(g, \phi) - e^*(g^{-S}, \phi) \mid S \subset N, |S| = s\}.$$

Of course, this is equivalent to minimizing the aggregate crime in the network g^{-S} that results from the removal of a set S of criminals. Given that aggregate crime increases with the aggregate network Bonacich centrality, the planner's problem becomes:

$$\min\{\beta(g^{-S}, \phi) \mid S \subset N, |S| = s\}. \tag{9}$$

When $s = 1$, the planner's problem (9) is equivalent to maximizing $\theta_i(g)$, $i \in N$. Indeed, $\beta(g^{-i}, \phi) = \beta(g, \phi) - \theta_i(g)$. Suppose now that $s = 2$. Reiterating this formula, it is plain to check that (9) is equivalent to solving:[28]

$$\max\{\theta_{i_1}(g) + \theta_{i_2}(g^{-i_1}) + \theta_{i_3}(g^{-i_1-i_2}) + ... + \theta_{i_s}(g^{-i_1-...-i_{s-1}}) \mid \{i_1, ..., i_s\} \subseteq N\}, \tag{10}$$

where $i_1, ..., i_s$ are different two by two. In words, the key group maximizes the sum of the individual inter-centrality measures of its members across the networks obtained through sequential removal of these members.[29] The idea behind this expression is the following. We must eliminate a set of players $S = \{i_1, ..., i_s\}$ in order to minimize the total number of (weighted) walks in the network, $\beta(g^{-S}, \phi)$. After deleting player i_1, the resulting number of paths is $\beta(g, \phi) - \theta_{i_1}(g)$. Now, the expression $\theta_{i_2}(g^{-i_1})$ counts the number of walks that touch i_2 *once player* i_1 *has been eliminated*, so that we are not counting the same path twice. Thus, $\beta(g, \phi) - \theta_{i_1}(g) - \theta_{i_2}(g^{-i_1})$ is the remaining set of walks after eliminating players i_1 and i_2, keeping in mind that we only want to count each walk once. By the previous argument, also note that the remaining set of weighted paths is the same if we change the order of deletion of these two players, that is:

$$\beta(g, \phi) - \theta_{i_1}(g) - \theta_{i_2}(g^{-i_1}) = \beta(g, \phi) - \theta_{i_2}(g) - \theta_{i_1}(g^{-i_2})$$

Extending this argument to the rest of the players in S, we obtain the expression for the number of paths after deleting all players in S:

$$\beta(g^{-S}, \phi) = \beta(g, \phi) - \theta_{i_1}(g) - \theta_{i_2}(g^{-i_1}) - \theta_{i_3}(g^{-i_1-i_2}) - ... - \theta_{i_s}(g^{-i_1-...-i_{s-1}})$$

so that, minimizing $\beta(g^{-S}, \phi)$ is equivalent to (10).

[28] As shown in the proof of Proposition 7. A heuristic argument follows after the proposition is stated.

[29] Note that this sum is independent of the order in which nodes are removed.

Group inter-centrality In what follows, we provide *a direct characterization of the key group on the original network* g that dispends with computing the nested sequence of networks resulting from sequential node removal. The key group characterization relies on a generalization of the inter-centrality network measure for groups. Given that individual inter-centrality captures both direct and indirect Bonacich-centrality measures, the generalization to a group of the inter-centrality measure needs to account for all the cross-countributions that arise both within and outside the group.

Consider some subset $S \subset N$ of criminals, $S \neq \emptyset$. Denote by $\beta_{i,S}(g, \phi)$ the contribution of this subset S to the centrality of any individual player $i \notin S$ outside this set in the network g. This is equal to:

$$\beta_{i,S}(g, \phi) = \sum_{k \geq |S|} \phi^k \beta^k_{i,S}(g),$$

where $\beta^k_{i,S}(g)$ counts the number of paths in g of length k starting from i and that go through all elements in S at least once. Note that this quantity is well-defined for low enough values of ϕ. We define $\beta^k_{i,S}(g)$ recursively as follows:

$$\beta^k_{i,\{j\}}(g) = \beta^k_{ij}(g), \text{ for all } i \neq j,$$

and

$$\beta^k_{i,S}(g) = \sum_{j \in S} \sum_{p=1}^{k} g^{[p]}_{ij} \beta^{k-p}_{j,S\setminus\{j\}}(g), \text{ for all } |S| > 1 \text{ and } i \notin S.$$

By convention, we set $\beta_{i,\emptyset}(g, \phi) = \beta_i(g, \phi)$.

Consider now the following expression:

$$\xi_S(g, \phi) = \sum_{i \in S} \beta_{i,S\setminus\{i\}}(g, \phi) + \sum_{i \notin S} \beta_{i,S}(g, \phi).$$

This formula counts the number of weighted walks in the network g that go through *all* the elements in S at least once. The expression adds up the walks stemming from nodes outside the set and those starting from nodes inside the set. In particular, specializing to singletons, we get $\xi_i(g, \phi) = \theta_i(g, \phi)$, that is, the individual inter-centrality measure of player i.

We now define the inter-centrality of a set S of players.

Definition 2 *For all $S \subset N$, $S \neq \emptyset$, let $\theta_S(g, \phi) = \sum_{\Omega \subseteq S} (-1)^{|\Omega|+1} \xi_\Omega(g, \phi)$.*

Note that, when S is a singleton, this definition coincides with the individual inter-centrality measure $\theta_i(g, \phi)$. More generally, when $|S| > 1$, the intercentrality of S is equal to:

$$\theta_S(g, \phi) = \theta_{i_1}(g) + \theta_{i_2}(g^{-i_1}) + \theta_{i_3}(g^{-i_1-i_2}) + ... + \theta_{i_s}(g^{-i_1-...-i_{s-1}}),$$

for any given labelling $\{i_1, ..., i_s\}$ of elements in the subset S.

A geometric characterization of the key group The following result now characterizes key groups of a given size s.

Proposition 7 *Let $1 \leq s \leq n-1$. A group S^* of size s is the key group that solves (9) if and only if S^* is a group with the highest group inter-centrality in g, that is, $S^* \in \arg\max\{\theta_S(g, \phi) \mid S \subset N, |S| = s\}$.*

This proposition provides a geometric characterization of the solution to (9) that generalizes the geometric characterization of the key player (see Proposition 4) to key groups of arbitrary size.

We now illustrate this result with an example.

Consider the network with eleven players in Figure 2, and consider the case where the key group size is $s = 2$. The next table shows the values of $\theta_S(g, \phi)$ for each possible subset S of size two when $\phi = 0.2$. For the sake of simplicity, subsets that yield the same network when they are removed are considered as equivalent:

Removed Group S	$\theta_S(g, \phi)$
$\{2, 7\}^*$	67.22
$\{2, 8\}$	64.01
$\{3, 8\}$	59.39
$\{1, 2\}$	56.66
$\{2, 6\}$	50.41
$\{2, 3\}$	46.96
$\{3, 4\}$	42.15

The key group is $\{2, 7\}$, that is, a set of two maximally connected nodes (with five direct contacts each), both connected to the central player 1, and each at a different side of this central player. This subset solves the following optimization problem:

$$\max_{i,j} \theta_{\{i,j\}}(g) = \max_{i,j} [\theta_i(g) + \theta_j(g^{-i})]$$

Suppose that we were to approximate the solution to this optimization problem with some greedy heuristics that pick up sequentially the player that maximizes the individual inter-centrality at each step. Formally, let

$$i_1^* = \arg\max_{i \in N} \theta_i(g)$$

and then, at each step $t \leq s$, choose the player i_t^* with maximum inter-centrality in the network where the previous players have been deleted, that is,

$$i_t^* \in \arg\max_{i \in N \setminus \{i_1^*, \ldots, i_{t-1}^*\}} \theta_i(g^{-\{i_1^*, \ldots, i_{t-1}^*\}})$$

breaking possible ties arbitrarily. This greedy algorithm first eliminates player 1, and then any other remaining player. Thus, the algorithm returns a group which is far from being optimal: there are many other groups that are better candidates than $\{1,2\}$. Indeed, in this example, player 1 is not only very central, but also its contribution to the inter-centrality of others is large. Hence, being greedy and eliminating it at the first stage reduces the chance of finding highly central players at further stages. And, in fact, player 1 is not part of the key group when $\phi = 0.2$.

A straightforward generalization of Lemma 1 establishes that the key group S^* solves $\arg\min\{n(g^{-S}) \mid |S| = s\}$ when ϕ is small enough. That is, for low values of ϕ, the key group minimizes the total number of remaining links (when the key group has been removed). Note, here, that $n(g^{-\{2,7\}}) \leq n(g^{-\{i,j\}})$, for all i, j, and this alternative characterization thus applies for $\phi = 0.2$.[30]

4 Joining crime networks

4.1 Equilibrium networks

The endogenous crime network game So far, we have assumed that the crime network was given. In some cases, though, criminals may have opportunities outside the crime network. For instance, petty delinquents may consider to enter the labor market and give up criminal activities. Here, we expand the model and endogeneize the crime network by allowing criminals to take a binary decision on whether to stay in the crime network or to drop out of it.[31] Formally, we consider the following two-stage game.

Fix an initial network g connecting agents.

In the first stage, each agent $i = 1, ..., n$ decides to enter the labor market or to become a criminal. This is a simple binary decision. These decisions are simultaneous. Let $c_i \in \{0,1\}$ denote i's decision, where $c_i = 1$ (resp. $c_i = 0$) stands for becoming a criminal (resp. entering the labor market), and denote by $\mathbf{c} = (c_1, ..., c_n)$ the corresponding population binary decision profile. We assume that agents entering the labor market earn a fixed wage $w > 0$. The payoff for criminals is determined in the second stage of the game.

In the second stage, criminals in $C(\mathbf{c}) = \{i \in \{1, ..., n\} \mid c_i = 1\}$ decide their crime effort level $e_i(\mathbf{c}) > 0$ that depends on the first-stage outcome $\mathbf{c}$. Given a crime effort profile $\mathbf{e}(\mathbf{c}) = [e_i(\mathbf{c})]_{i \in C(\mathbf{c})}$, the individual expected crime gains are equal to:

$$u_i(\mathbf{e}(\mathbf{c}), g) = y_i(\mathbf{e}(\mathbf{c})) - p_i(\mathbf{e}(\mathbf{c}), g)f$$

Equilibrium networks Let $\mathbf{c} = (c_1, ..., c_n) \in \{0,1\}^n$ be a population binary decision profile, and $C(\mathbf{c})$ the corresponding set of active criminals, with cardinality $c(\mathbf{c})$. Assume that $C(\mathbf{c}) \neq \emptyset$.

[30] Recall that, for the case of a single player removal, this alternative characterization of the key player (namely, minimizing the total number of remaining links) applies when $\phi = 0.1$ but not anymore when $\phi = 0.2$.

[31] See Antoni Calvó-Armengol and Matthew O. Jackson (2004) for a similar endogenous game of network formation in the context of the labor market, where the binary decision for agents is to enter the labor market network or to drop out.

The network that connects active criminals is determined by the collection of individual drop in decisions $\mathbf{c}$. We denote this network by $g(\mathbf{c})$. This is the network induced by the original network g on the set of active criminals $C(\mathbf{c})$. Two criminals $i, j \in C(\mathbf{c})$ are directly linked in $g(\mathbf{c})$ if and only if a direct link between them pre-exists in g. Formally, $g_{ij}(\mathbf{c}) = g_{ij}c_i c_j$.

We denote by $\mathbf{G}(\mathbf{c}) = [g_{ij}(\mathbf{c})]_{i,j\in C(\mathbf{c})}$ the reduced adjacency matrix corresponding to this network. By definition, this is a $c(\mathbf{c})-$dimensional matrix.[32]

Denote by $\mathbf{I}(\mathbf{c})$ the identity matrix of size $c(\mathbf{c})$, by $\mathbf{1}(\mathbf{c})$ the $c(\mathbf{c})-$dimensional column vector of ones, and by $\mathbf{J}(\mathbf{c}) = \mathbf{1}(\mathbf{c})\cdot\mathbf{1}^T(\mathbf{c})$ the $c(\mathbf{c})-$dimensional square matrix of ones. From Proposition 1, the Nash equilibrium $\mathbf{e}^*(\mathbf{c})$ of the second-stage game following the first-stage decision $\mathbf{c}$ is the unique solution to the following matrix equation:

$$[\mathbf{J}(\mathbf{c}) + \mathbf{I}(\mathbf{c}) - \phi\mathbf{G}(\mathbf{c})]\cdot\mathbf{e}(\mathbf{c}) = \frac{1-\pi}{\gamma}\mathbf{1}(\mathbf{c})$$

Following Proposition 3, this unique Nash equilibrium is given by:[33]

$$\mathbf{e}^*(\mathbf{c}) = \frac{1-\pi}{\gamma}\frac{1}{1+\beta(g(\mathbf{c}),\phi)}\boldsymbol{\beta}(g(\mathbf{c}),\phi). \tag{11}$$

We now provide a general characterization of the subgame perfect equilibria of the full game. We first introduce some useful notations.

For all $\mathbf{c}, \mathbf{c}' \in \{0,1\}^n$, the *join* of $\mathbf{c}$ and $\mathbf{c}'$, denoted by $\mathbf{c}\vee\mathbf{c}'$, is the binary population profile defined by $(\mathbf{c}\vee\mathbf{c}')_i = \max\{c_i, c_i'\}$, for all $i = 1, ..., n$. In words, $\mathbf{c}\vee\mathbf{c}'$ "adds up" the criminal decisions in $\mathbf{c}$ and $\mathbf{c}'$. In particular, $C(\mathbf{c}\vee\mathbf{c}') = C(\mathbf{c})\cup C(\mathbf{c}')$.

Let $\boldsymbol{\nu}^1, ..., \boldsymbol{\nu}^n$ be the canonical base of $\{0,1\}^n$, where the coordinates of $\boldsymbol{\nu}^i$ are all zeros except a one in the ith position. For instance, $\boldsymbol{\nu}^1 = (1,0,...,0)^T$, $\boldsymbol{\nu}^2 = (0,1,0,...,0)^T$ and $\boldsymbol{\nu}^n = (0,...0,1)^T$. Then, $C(\mathbf{c}\vee\boldsymbol{\nu}^i) = C(\mathbf{c})\cup\{i\}$. In words, the set of criminals in $\mathbf{c}\vee\boldsymbol{\nu}^i$ is deduced from that in $\mathbf{c}$ by adding agent i to the active crime pool.

At the subgame perfect equilibria $(\mathbf{c}^*, \mathbf{e}^*(\cdot))$ of the full game, the payoffs to workers are equal to w while the payoffs accruing to criminals are given by the Nash equilibrium strategies of the second-stage game as in (11). It is readily checked that the Nash equilibrium payoffs for active criminals at the second-stage game are equal to the square of their crime efforts, that is,[34]

$$u_i(\mathbf{e}^*(\mathbf{c}^*), g) = \gamma e_i^{*2}(\mathbf{c}), \text{ for all } i \in C(\mathbf{c}).$$

[32] The adjacency matrix $[g_{ij}(\mathbf{c})]_{1\leq i,j\leq n}$ is a square matrix of size n. $\mathbf{G}(\mathbf{c})$ is obtained from this matrix by eliminating $n - c(\mathbf{c})$ rows and columns of 0s. It is thus of reduced size $c(\mathbf{c})$.

[33] Whenever ϕ is smaller or equal than the reciprocal of the highest eigenvalue of $\mathbf{G}(\mathbf{c})$.

[34] Indeed, we deduce from $\partial u_i(\mathbf{e}^*(\mathbf{c}^*))/\partial e_i = 0$ that $\gamma e_i^*(\mathbf{c}) = 1 - \pi - \gamma\sum_{j\in C(\mathbf{c})}[1-\rho g_{ij}(\mathbf{c})]e_j^*(\mathbf{c})$, for all $i \in C(\mathbf{c})$. After some manipulation,

$$u_i(\mathbf{e}^*(\mathbf{c}^*), g) = e_i^*(\mathbf{c})[1-\pi-\gamma\sum_{j\in C(\mathbf{c})}[1-\rho g_{ij}(\mathbf{c})]e_j^*(\mathbf{c})] = \gamma e_i^{*2}(\mathbf{c}).$$

At the equilibria of the full game, no unilateral deviation is profitable, that is, no worker gains by becoming a criminal, nor does any criminal gain by becoming a worker.

Proposition 8 *There exists a unique* $0 < \widetilde{\phi} \leq 1$ *such that, for all* $0 \leq \phi < \widetilde{\phi}$*, the binary decision profile* $\mathbf{c}^* \in \{0,1\}^n, \mathbf{c}^* \neq \mathbf{0}$ *is part of a subgame perfect equilibrium of the full game if and only if:*[35]

$$\begin{aligned} \frac{\beta_i(g(\mathbf{c}^*),\phi)}{1+\beta(g(\mathbf{c}^*),\phi)} &\geq \frac{\sqrt{\gamma w}}{1-\pi}, \text{ for all } i \in C(\mathbf{c}) \\ \frac{\beta_i(g(\mathbf{c}^* \vee \boldsymbol{\nu}^i),\phi)}{1+\beta(g(\mathbf{c}^* \vee \boldsymbol{\nu}^i),\phi)} &< \frac{\sqrt{\gamma w}}{1-\pi}, \text{ for all } i \notin C(\mathbf{c}) \end{aligned}$$

The endogenous crime networks are thus characterized by a set of inequalities.

Existence of multiple equilibrium crime pools is illustrated in Figure 3 that draws the correspondence between w and the total crime e^* for all corresponding subgame perfect equilibria for the network in Figure 2 when $\phi = 0.2$. Note that, as the wage rate w increases, the set of active criminals tends to shrink, producing a decreasing trend in the total crime effort. For instance, when $w = 0.004$, there are three possible equilibria. The first is "large" and consists of the whole network with eleven players, with a resulting total crime equal to 0.7914. The second and the third are two equivalent "small" equilibria where either the fully connected five players on the left side, or the ones on the right side, are the active criminals, with a total amount of crime equal to 0.7785 in either cases.

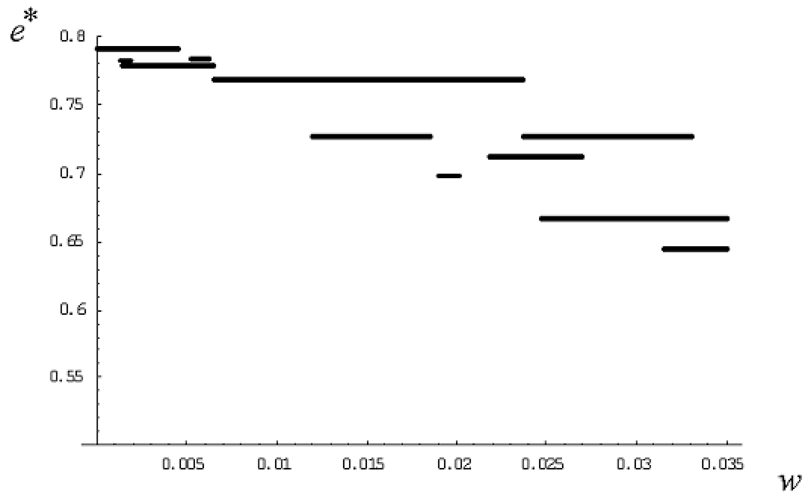


Figure 3: Equilibrium correspondence of the two-stage game when $\phi = 0.2$.

[35] Note that this characterization implies that, when a player is indifferent between becoming a criminal or a worker, we assume that he becomes a criminal. This is without loss of generality.

4.2 Finding the key player

Given that this game usually displays multiple subgame perfect equilibria in the endogenous crime network game, we define $e^*(g, w, \phi)$ to be the maximum aggregate equilibrium crime level when the initial population network is g, the labor market wage is w and the deterrence effort is ϕ. That is, this is equal to the total amount of crime in the worst case scenario of maximum deliquency.

Consider some binary decision profile $\mathbf{c}$. Let i be an active criminal, that is $c_i = 1$. Suppose that criminal i switches his current decision to $c_i = 0$, that is, criminal i drops out from the crime pool and enters the labor market instead. The binary decision profile then becomes $\mathbf{c} - \boldsymbol{\nu}^i$, and the new set of active criminals is $C(\mathbf{c} - \boldsymbol{\nu}^i) = C(\mathbf{c})\backslash\{i\}$. The drop out of criminal i from the crime pool also alters the network structure connecting active criminals, as any existing direct link between i and any other criminal in $C(\mathbf{c})$ is removed. The new network connecting active criminals is then $g(\mathbf{c})^{-i} = g(\mathbf{c} - \boldsymbol{\nu}^i)$, and the aggregate crime level becomes:

$$e^*(\mathbf{c} - \boldsymbol{\nu}^i) = \frac{1-\pi}{\gamma} \frac{\beta(g(\mathbf{c} - \boldsymbol{\nu}^i), \phi)}{1 + \beta(g(\mathbf{c} - \boldsymbol{\nu}^i), \phi)}$$

The key player problem acquires a different shape in the setting with endogenous formation of crime pools. Initially, the planner must choose a player to remove from the network. Then, players play the two-stage crime game. First, they decide whether to enter the crime pool or not. Second, criminals choose how much effort to exert. In this context, there is an added difficulty to the planner's decision. The removal of a player from the network affects the rest of the players' decisions to become active criminals. And this fact should be taken into account by the planner in order to attain an equilibrium with minimum total crime. The right choice of the key player should be based upon the resulting crime pool that will result from that decision, that is, what the remaining players will decide regarding their criminal activities.

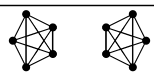
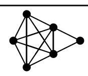
We show, with the help of an example, that there is no trivial geometric recipe for the key player problem in this case.

Consider again the network in Figure 2 with eleven players. Recall that, when $\phi = 0.2$ and the network of criminals is exogenously fixed (or, equivalently, the outside option is $w = 0$), the key player was the player acting as a bridge, player 1. If we consider endogenous crime network formation in the two-stage game, the results may differ from the previous case. For low wages, player 1 is also the key player and the resulting equilibrium network is the whole remaining network, that is, an equilibrium with ten criminals split into two fully connected cliques of five criminals. When w is higher, though, type-2 criminals become the key player[36] and the equilibrium network now encompasses six different players. It consists of a clique of five fully intraconnected players together with player 1.

These results are summarized in the following table, that gives, for two different values of w, the key player, the highest aggregate crime that results from eliminating this key player, and the

[36] In fact, any player except player 1 is the key player for $w = 0.003$.

equilibrium crime network.

	$w = 0.001$	$w = 0.003$
$e^*(g^{-1}, w, \phi)$	0.7843	0.7843
$e^*(g^{-2}, w, \phi)$	0.7847	0.7785
Key Player	1	2
Final Crime pool		

Intuitively, when outside opportunities are high enough, all players from the same side of the player being removed do not have enough incentives to enter the crime pool at the first stage of the game. Hence, we do not get a "large" equilibrium with many players, and this constitutes an advantage for the planner, that will choose to delete node 2. This example implicitly explains how one policy (providing a higher w) increases the effectiveness of another policy (choosing the key player) in order to reduce crime. These policies are complementary from the point of view of their effects on total crime, although we are aware that they may be substitutes from the natural point of view of a budget restricted planner who has to implement costly policies.

5 Concluding discussion

Although we focus in this paper on crime outcomes, our analysis sheds light on a more general non-cooperative exploration of the network structure of local externalities or peer effects, and on the optimal design of network-based policies in this context. Two examples where our methodology could be applied are models of local public goods, as in Yann Bramoullé and Rachel Kranton (2004), and bilateral firm collaborations in research and development activities, as in Sanjeev Goyal and José-Luis Moraga (2001). We now sketch the general features of our methodology.

Consider a finite population game on a network where players take their decisions on one segment of the real line. Suppose that individual best responses are linear in players' actions, and can be additively decomposed into a global and a local component. The global component is common to all players' best responses. The local component varies across players with their available set of contacts, and reflects the network embeddedness of each player.

Suppose further that individual decisions are strategic substitutes. Here, strategic substitutability is equivalent to a negative net effect of the marginal global and local components of the best response. In particular, if the partial derivatives of both effects are negative, strategic substitutability follows trivially. If these partial derivatives have, instead, different signs, as in the crime decision game analyzed here, strategic substitutability only holds when the relative values of the global and the local marginal effects are suitably ranked.

For this type of games, it is a straightforward extension of Propositions 1 and 3 to establish (generic) uniqueness of an interior Nash equilibrium characterized by a Bonacich-type index. In words, for a whole class of non-cooperative games with local externalities or peer effects, and for arbitrary network structures for these local externalities or peer effects, equilibrium outcomes boil down to a closed-form network centrality measure. This is true for a rich variety of network structures, including directed networks, for which $g_{ij} = g_{ji}$ does not necessarily hold, and weighted networks with arbitrary link intensities, for which g_{ij} can take any value on the real line.[37] These results hold under a simple condition relating the payoff structure to the network topology, namely, the ratio of the global versus local marginal effects be lower (in absolute value) than the inverse of the highest eigenvalue of the network adjacency matrix.

The equilibrium analysis described above opens the door to a more general policy analysis of the key player problem. Consider a population of $n+1$ agents indexed by $i = 0, 1, ..., n$ and connected by a network g. Suppose that the planner holds the outcome of some targeted player to some fixed exogenous value $s \in \mathbb{R}$. The case $s > 0$ (resp. $s < 0$) can be interpreted as a subsidy (resp. tax), while $s = 0$ corresponds to the key player problem solved above. Suppose, without loss of generality, that the targeted player is $i = 0$. The remaining players $i = 1, ..., n$ play an $n-$player game on the network g^{-0}. We denote by $\mathbf{e}^*_{-0}(g, s)$ the Nash equilibrium of this game, and by $e^*_{-0}(g, s)$ the corresponding aggregate outcome. We keep the same notations for the payoff function than in the crime decision game.

Given an $n-$dimensional vector $\mathbf{v} = (v_1, ..., v_n)$, define the weighted Bonacich centrality measure on a network g with weights $v_1, ..., v_n$ by $\boldsymbol{\beta}_{\mathbf{v}}(g, \phi) = [\mathbf{I}-\phi\mathbf{G}]^{-1}\cdot\mathbf{v}$. The standard Bonacich centrality measure corresponds to uniform unitary weights $\mathbf{1}$. Denote by $\mathbf{g_0}$ the $n-$dimensional column vector with coordinates $g_{01}, ..., g_{0n}$ that keeps track of player 0's direct contacts in g, and let $\tau = (1-\pi)/\gamma$. Then, the total population output at equilibrium is $s + e^*_{-0}(g, s)$, where:

$$e^*_{-0}(g, s) = \frac{1}{1 + \beta(g^{-0}, \phi)}[(\tau - s)\beta(g^{-0}, \phi) + \phi s \beta_{\mathbf{g_0}}(g^{-0}, \phi)].$$

Given an objective function related to the total population output $s + e^*_{-0}(g, s)$, and a set of constraints, the planner's problem is to fix optimally the value of s of the subsidy and the identity i of the optimal target for this subsidy. Holding s constant, the choice of the optimal target is a simple finite optimization problem. In particular, when $s = 0$, the solution to this problem is $i^* \in \arg\max \theta_i(g)$, that is, the player with the highest inter-centrality measure.

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Appendix

Proof of Proposition 1: First, note that $\partial^2 u_i(\mathbf{e})/\partial e_i^2 = -2\gamma < 0$. Therefore, if an interior equilibrium exists, it is given by the unique solution to:

$$\frac{\partial u_i(\mathbf{e})}{\partial e_i} = 1 - \pi - \gamma e_i - \gamma \sum_j e_j + \pi\lambda \sum_j g_{ij} e_j = 0.$$

This is an $n-$dimensional linear system that we can write in matrix form:

$$[\mathbf{J} + \mathbf{I} - \phi\mathbf{G}] \cdot \mathbf{e} = \frac{1-\pi}{\gamma}\mathbf{1}.$$

Define $\mathbf{M}(g,\phi) = \mathbf{J} + \mathbf{I} - \phi\mathbf{G}$, and denote by $\det[\mathbf{M}(g,\phi)]$ its determinant. We show that there exists some finite set $\mathcal{Z} \in \mathbb{R}$ such that $\det[\mathbf{M}(g,\phi)] \neq 0$, for all $\phi \notin \mathcal{Z}$ and for all g on $\{1,...,n\}$.

Consider some network g. It is readily checked that $\det[\mathbf{M}(g,\phi)]$ is a polynomial in ϕ of degree smaller than n.[38] Therefore, $\det[\mathbf{M}(g,\phi)]$ has at most n different roots $\left\{\widetilde{\phi}_1(g), \ldots, \widetilde{\phi}_m(g)\right\}$, $m \leq n$, such that $\det[\mathbf{M}(g,\widetilde{\phi}_i(g))] = 0$ for all $1 \leq i \leq m$. Given that there are exactly $2^{\frac{n(n-1)}{2}}$ different networks g on $\{1,...,n\}$, the set of values $\mathcal{Z}$ of ϕ such that $\det[\mathbf{M}(g,\phi)] = 0$ for some g is finite, with $|\mathcal{Z}| \leq n 2^{\frac{n(n-1)}{2}}$.

Next, when $\pi = 0$, the unique solution to (3) is $(n+1)e_i^*\gamma = 1$. Suppose that $\gamma \geq \lambda$. Then, $n\overline{e}\gamma = 1$, implying that $e_i^* \in (0,\overline{e})$. By continuity, there exists $0 < \varepsilon(g) \leq 1$ such that the solutions to this system of equations are non-negative for all $\pi \in (0, \varepsilon(g))$. Let g^N such that $g_{ij}^N = 1$, for all $i \neq j$. Let $\overline{\pi} = \min\left\{\varepsilon(g) \mid g \subseteq g^N\right\}$. Let $\overline{\phi} = \overline{\pi}\lambda/\gamma$. Then, $0 < \overline{\phi} \leq 1$, and for all $0 \leq \phi < \overline{\phi}$, the solutions to (3) are all non-negative for all g on N. ∎

Proof of Proposition 2. Given that $\mathbf{I}, \mathbf{J}, \mathbf{G}$ are all symmetric matrices, $\mathbf{M}(g,\phi) = \mathbf{I} + \mathbf{J} - \phi\mathbf{G}$ is also symmetric. Denote by $\mathbf{e}^*(g)$ the unique solution to (3), that is, $\mathbf{M}(g,\phi) \cdot \mathbf{e}^*(g) = (1-\pi)/\gamma\mathbf{1}$.

Suppose that g and g' are adjacent networks with $g \subset g'$. Without loss of generality, let $g' = g \cup \{12\}$. Then,

$$\mathbf{M}(g,\phi) = \mathbf{M}(g',\phi) + \phi \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 \\ 1 & 0 & 0 & \ldots & 0 \\ 0 & 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 0 \end{bmatrix}.$$

Premultiplying (3) by $\mathbf{e}^{*T}(g')$ and noting that $\mathbf{e}^{*T}(\mathbf{g}') \cdot \mathbf{M}(g') = (1-\pi)/\gamma\mathbf{1}^T$, we have:

$$\begin{aligned} \mathbf{e}^{*T}(g') \cdot \mathbf{M}(g) \cdot \mathbf{e}^*(g) &= (1-\pi)/\gamma \mathbf{e}^{*T}(g') \cdot \mathbf{1} \\ &= (1-\pi)/\gamma \mathbf{1}^T \cdot \mathbf{e}^*(g) + \phi[e_2^*(g')e_1^*(g) + e_1^*(g')e_2^*(g)] \end{aligned}$$

[38] Indeed, $\det[\mathbf{M}(\rho,g)]$ is a polynomial of highest degree in ρ when $g_{ij} = 1$ for all $i \neq j$, in which case $\det[\mathbf{M}(\rho,g)] = (1+\rho)^n + n(1-\rho)(1+\rho)^{n-1}$, which is a polynomial in ρ of degree exactly n.

Hence, the total crime level under g', is higher than the total crime level under g, that is,

$$\mathbf{e}^{*T}(g') \cdot \mathbf{1} > \mathbf{1}^T \cdot \mathbf{e}^*(g)$$

when g and g' are adjacent networks. The inequality extends to any two nested networks by iterative application of the inequality along any chain of adjacent networks between them. ∎

Proof of Proposition 3. Given a crime effort profile $\mathbf{e}$, denote by $e = \mathbf{1}^T \cdot \mathbf{e}$ the total crime effort exerted by criminals. Noting that $\mathbf{J} = \mathbf{1} \cdot \mathbf{1}^T$, (3) can be written as:

$$[\mathbf{I} - \phi\mathbf{G}] \cdot \mathbf{e} = (\frac{1-\pi}{\gamma} - e)\mathbf{1}.$$

Suppose now that $\mathbf{I} - \phi\mathbf{G}$ is invertible. Then, we get:

$$\mathbf{e} = (\frac{1-\pi}{\gamma} - e)[\mathbf{I} - \phi\mathbf{G}]^{-1} \cdot \mathbf{1} = (\frac{1-\pi}{\gamma} - e)\boldsymbol{\beta}(g,\phi).$$

Premultiplying by $\mathbf{1}^T$ and manipulating terms, we get:

$$e = \frac{1-\pi}{\gamma} \frac{\beta(g,\phi)}{1+\beta(g,\phi)}.$$

Plugging back into the first equality, we deduce the result.

Now, when ϕ is smaller or equal than the reciprocal of the largest eigenvalue of the adjacency matrix $\mathbf{G}$ of the network g, $\mathbf{I} - \phi\mathbf{G}$ is invertible and can be written as a Taylor expansion. Let $n(g) = \sum_{i,j} g_{ij}$ denote the number of links in g (counted twice), $\delta(g) = \min \left\{ \sum_{j=1}^n g_{ij} \mid i = 1, ..., n \right\}$ be the minimum degree or number of direct links for the nodes of g, and $\mu_0(g)$ be the largest eigenvalue of $\mathbf{G}$. Then, we have (Kunfu Fang *et al.* 2001):

$$\mu_0(g) \leq \frac{1}{2}[\delta(g) - 1 + \sqrt{(\delta(g)+1)^2 + 4\left[n(g) - n\delta(g)\right]}],$$

and this upper bound, denoted by $\overline{\mu}(g)$, is sharp as equality is obtained, e.g., when either g is a regular network, that is, $\sum_{j=1}^n g_{ij} = \delta(g)$, for all i, or when g is a network where degrees take on only two possible values. Then, $\boldsymbol{\beta}(g,\phi)$ is well-defined whenever $0 \leq \phi \leq 1/\overline{\mu}(g)$. ∎

Proof of Proposition 4. We compute $\beta(g^{-i},\phi)$. For all $j \neq i$, (8) implies that:

$$\beta_j(g^{-i},\phi) = \beta_j(g,\phi) - \beta_{ji}(g,\phi).$$

Therefore,

$$\begin{aligned} \beta(g^{-i},\phi) &= \sum_{j\neq i} \beta_j(g^{-i},\phi) = \sum_{j\neq i} \beta_j(g,\phi) - \sum_{j\neq i} \beta_{ji}(g,\phi) \\ &= \sum_{j\neq i} \beta_j(g,\phi) + \beta_i(g,\phi) - \theta_i(g,\phi) = \beta(g,\phi) - \theta_i(g,\phi), \end{aligned}$$

and the result follows. ∎

Proof of Lemma 1. The Bonacich's centrality measure vector is defined as:

$$\boldsymbol{\psi}(g, \alpha, \beta) = \alpha(\mathbf{I} - \beta\mathbf{G})^{-1}\mathbf{G} \cdot \mathbf{1}$$

Given the relation between $\boldsymbol{\psi}(g, \alpha, \beta)$ and our variation $\boldsymbol{\beta}(g, \phi)$:

$$\boldsymbol{\beta}(g, \phi) = \mathbf{1} + \phi\boldsymbol{\psi}(g, 1, \phi)$$

it is obvious that the total centrality $\beta(g, \phi)$ is given by

$$\beta(g, \phi) = n + \phi \sum_{i=1}^{n} \psi_i(g, 1, \phi)$$

Given the fact, that $\psi_i(g, 1, \phi) \to n_i(g)$ as $\phi \to 0$, the result follows. ∎

Proof of Proposition 5. By definition of $\mathbf{M}(g, \phi)$, we have $[\mathbf{I} - \phi\mathbf{G}] \cdot \mathbf{M}(g, \phi) \cdot \boldsymbol{\nu}^i = \boldsymbol{\nu}^i$, where:

$$\mathbf{M}(g, \phi) \cdot \boldsymbol{\nu}^i = \begin{bmatrix} m_{1i}(g, \phi) \\ \vdots \\ m_{ii}(g, \phi) \\ \vdots \\ m_{ni}(g, \phi) \end{bmatrix}.$$

The i's row of the previous vectorial equality leads to:

$$m_{ii}(g, \phi) = 1 + \phi \sum_j g_{ij} m_{ij}(g, \phi). \tag{12}$$

Throughout we use the symmetry of $\mathbf{M}(g, \phi)$, inherited from that of $\mathbf{G}$.

For all $k \neq i$, define $s_i^k(g, \phi) = \sum_j g_{kj} m_{ij}(g, \phi)$. Given that $m_{ij}(g, \phi)$ counts the number of weighted paths in g that start from i and that end at j, and given that $g_{kj} = 1$ if and only if k and j are directly linked to one another, $s_i^k(g, \phi)$ can be interpreted as the number of weighted paths in g that start from i and that end up in the direct neighborhood of player k. Similarly, define $\widetilde{s}_i^k(g^{-k}, \phi) = \sum_j g_{kj} m_{ij}(g^{-k}, \phi)$. This is the number of weighted paths in g that start from i and that end up in the direct neighborhood of player k but that do not include k. In particular, it is clear that:

$$m_{ij}(g, \phi) = m_{ij}(g^{-k}, \phi) + \phi\widetilde{s}_i^k(g^{-k}, \phi) m_{kj}(g, \phi).$$

Summing over the neighborhood of player k leads to:

$$\sum_j g_{jk} m_{ij}(g^{-k}, \phi) = \sum_j g_{jk} m_{ij}(g, \phi) - \phi\widetilde{s}_i^k(g^{-k}, \phi) \sum_j g_{jk} m_{kj}(g, \phi),$$

equivalent to:

$$\widetilde{s}_i^k(g^{-k},\phi) = s_i^k(g,\phi) - \phi \widetilde{s}_i^k(g^{-k},\phi) s_k^k(g,\phi) \Leftrightarrow \widetilde{s}_i^k(g^{-k},\phi) = \frac{s_i^k(g,\phi)}{1+\phi s_k^k(g,\phi)}.$$

But (12) implies that $1+\phi s_k^k(g,\phi) = m_{kk}(g,\phi)$. Therefore,

$$\widetilde{s}_i^k(g^{-k},\phi) = \frac{s_i^k(g,\phi)}{m_{kk}(g,\phi)}.$$

Plugging back into the expression for $m_{ij}(g,\phi)$ above gives:

$$\begin{aligned} m_{ij}(g^{-k},\phi) &= m_{ij}(g,\phi) - \phi \frac{m_{kj}(g,\phi)}{m_{kk}(g,\phi)} s_i^k(g,\phi) \\ &= m_{ij}(g,\phi) - \phi \frac{m_{kj}(g,\phi)}{m_{kk}(g,\phi)} \sum_j g_{kj} m_{ij}(g,\phi). \end{aligned}$$

Now, if we sum over all j we obtain:

$$\beta_i(g^{-k},\phi) = \beta_i(g,\phi) - \phi \frac{\beta_k(g,\phi)}{m_{kk}(g,\phi)} \sum_j g_{kj} m_{ij}(g,\phi),$$

and summing again over all $i \neq k$ we get:

$$\beta(g^{-k},\phi) = \beta(g,\phi) - \beta_k(g,\phi) - \phi \frac{\beta_k(g,\phi)}{m_{kk}(g,\phi)} \sum_j g_{kj} \sum_{i\neq k} m_{ij}(g,\phi).$$

But

$$\begin{aligned} \phi \sum_j g_{kj} \sum_{i\neq k} m_{ij}(g,\phi) &= \phi \sum_j g_{kj} (\beta_j(g,\phi) - m_{kj}(g,\phi)) \\ &= \phi \sum_j g_{kj} \beta_j(g,\phi) - \phi \sum_j g_{kj} m_{kj}(g,\phi). \end{aligned}$$

From $[\mathbf{I} - \phi \mathbf{G}] \cdot \boldsymbol{\beta} = \mathbf{1}$ we conclude that $\phi \sum_j g_{kj} \beta_j(g,\phi) = 1 + \beta_k(g,\phi)$. Then, using (12) we get:

$$\phi \sum_j g_{kj} \sum_{i\neq k} m_{ij}(g,\phi) = \beta_k(g,\phi) - m_{kk}(g,\phi).$$

Plugging this back into the expression for $\beta(g^{-k},\phi)$ gives:

$$\begin{aligned} \beta(g^{-k},\phi) &= \beta(g,\phi) - \beta_k(g,\phi) - \frac{\beta_k(g,\phi)}{m_{kk}(g,\phi)} (\beta_k(g,\phi) - m_{kk}(g,\phi)) \\ &= \beta(g,\phi) - \frac{\beta_k(g,\phi)^2}{m_{kk}(g,\phi)}. \end{aligned}$$

Now, recall from the previous proof that the intercentrality index of player k is given by $\theta_k(g,\phi) = \beta(g,\phi) - \beta(g^{-k},\phi)$, and the result follows. ∎

Proof of Proposition 6. Simple algebra leads to:

$$\eta_i(g,\phi) = \frac{\frac{\theta_i(g,\phi)}{1+\beta(g,\phi)-\theta_i(g,\phi)}}{\frac{1}{n}\sum_{j=1}^{n}\frac{\theta_j(g,\phi)}{1+\beta(g,\phi)-\theta_j(g,\phi)}}, \text{ for all } i = 1, ..., n.$$

By definition, $\theta_{i^*}(g,\phi) \geq \theta_i(g,\phi)$, for all $i = 1, ..., n$. This implies that:

$$\frac{1+\beta(g,\phi)-\theta_{i^*}(g,\phi)}{1+\beta(g,\phi)-\theta_j(g,\phi)} \leq 1, \text{ for all } j = 1, ..., n,$$

and, thus $\eta_{i^*}(g,\phi) \geq \theta_{i^*}(g,\phi)/\overline{\theta}(g,\theta)$. Noting that $\theta_{i^*}(g,\phi) \geq \overline{\theta}(g,\theta) + \sigma_\theta(g,\phi)$, we can conclude. ∎

Proof of Proposition 7. We first establish two useful Lemmata.

Lemma 2 *Suppose that* $n \geq 2$. *For all* $S \subset N$ *and for all* $i, j \notin S$, $\beta_{j,S}(g^{-i},\phi) = \beta_{j,S}(g,\phi) - \beta_{j,S\cup\{i\}}(g,\phi)$.

Proof. For all $i \in N$ and $\ell \geq 1$, let:

$$P_i^\ell(g) = \{\{i_0, ..., i_\ell\} \mid i_0 = i, i_{p+1} \in N, g_{i_p i_{p+1}} = 1, \forall 0 \leq p \leq \ell - 1\}.$$

This is the set of paths of length ℓ in g that start at i. We denote by ϱ a generic element of $P_i^\ell(g)$. This is an ordered list of (at least two) nodes in g where each node (except the first one) is directly linked to its predecessor. For all $S \subset N$, $S \neq \emptyset$, $i \notin S$ and $k \geq |S|$, where $|S|$ denotes the cardinality of S, we have:

$$\beta_{i,S}^k(g) = \left|\{\varrho \in P_i^k(g) \mid j \in S \Rightarrow j \in \varrho\}\right|.$$

Let $i, j \notin S$ and $k \geq |S|$. We have, $P_j^k(g^{-i}) = P_j^k(g)\backslash\{\varrho \in P_j^k(g) \mid i \in \varrho\}$. Therefore,

$$\beta_{j,S}^k(g^{-i}) = \beta_{j,S}^k(g) - \left|\{\varrho \in P_j^k(g) \mid i \in \varrho, j' \in S \Rightarrow j' \in \varrho\}\right|.$$

But,

$$\{\varrho \in P_j^k(g) \mid i \in \varrho, j' \in S \Rightarrow j' \in \varrho\} = \{\varrho \in P_j^k(g) \mid j' \in S \cup \{i\} \Rightarrow j' \in \varrho\},$$

implying that $\beta_{j,S}^k(g^{-i}) = \beta_{j,S}^k(g) - \beta_{j,S\cup\{i\}}^k(g)$. The result follows. *Q.E.D.*

Lemma 3 $\xi_S(g^{-i},\phi) = \xi_S(g,\phi) - \xi_{S\cup\{i\}}(g,\phi)$, *for all* $S \subset N$ *and all* $i \notin S$.

Proof. Let $S \subset N$ and $i \notin S$. By definition,

$$\xi_{S\cup\{i\}}(g) = \sum_{j\in S\cup\{i\}} \beta_{j,S\cup\{i\}\backslash\{j\}}(g,\phi) + \sum_{j\notin S\cup\{i\}} \beta_{j,S\cup\{i\}}(g,\phi).$$

By the Lemma above, this can be written as:

$$\begin{aligned}
\xi_{S\cup\{i\}}(g) &= \sum_{j\in S}[\beta_{j,S\backslash\{j\}}(g,\phi)-\beta_{j,S\backslash\{j\}}(g^{-i},\phi)]+\beta_{i,S}(g,\phi) \\
&\quad +\sum_{j\notin S}[\beta_{j,S}(g,\phi)-\beta_{j,S}(g^{-i},\phi)]-\beta_{i,S\cup\{i\}}(g,\phi) \\
&= \xi_S(g)-\xi_S(g^{-i})+\beta_{i,S}(g,\phi)-\beta_{i,S\cup\{i\}}(g,\phi).
\end{aligned}$$

But, by definition, $\beta_{i,S}(g,\phi)=\beta_{i,S\cup\{i\}}(g,\phi)$. The result follows *Q.E.D.*

To establish the proposition, we show that

$$\beta(g^{-S},\phi)=\beta(g,\phi)-\theta_S(g,\phi)=\beta(g,\phi)-\sum_{\Omega\subseteq S}(-1)^{|\Omega|+1}\xi_\Omega(g,\phi).$$

We establish the result by induction on the size of S. The case where $|S|=1$ is clear. Let $S\subset N$ such that $|S|\geq 2$, and suppose that the result is true for all S' such that $|S'|<|S|$.

Let $i\in S$. Note that $S\backslash\{i\}\neq\emptyset$. We have

$$\beta(g^{-S})=\beta(g^{-S\backslash\{i\}-i})=\beta(g^{-S\backslash\{i\}})-\theta_i(g^{-S\backslash\{i\}}).$$

By the induction hypothesis, this becomes:

$$\beta(g^{-S})=\beta(g,\phi)-\sum_{\Omega\subseteq S\backslash\{i\}}(-1)^{|\Omega|+1}\xi_\Omega(g,\phi)-\theta_i(g^{-S\backslash\{i\}}). \tag{13}$$

We now compute $\theta_i(g^{-S\backslash\{i\}})$. Recall that, by definition, $\theta_i(g^{-S\backslash\{i\}})=\xi_i(g^{-S\backslash\{i\}})$. Consider one ordered labelling of elements in S, that is, $S=\{i_1,...,i_s\}$, where $i_1=i$. The previous Lemma implies that:

$$\begin{aligned}
\xi_i(g^{-S\backslash\{i\}}) &= \xi_i(g^{-S\backslash\{i,i_2\}})-\xi_{\{i,i_2\}}(g^{-S\backslash\{i,i_2\}}) \\
&= \xi_i(g^{-S\backslash\{i,i_2,i_3\}})-\xi_{\{i,i_3\}}(g^{-S\backslash\{i,i_2,i_3\}})-\xi_{\{i,i_2\}}(g^{-S\backslash\{i,i_2\}}) \\
&= \xi_i(g^{-S\backslash\{i,i_2,i_3,i_4\}})-\xi_{\{i,i_4\}}(g^{-S\backslash\{i,i_2,i_3,i_4\}})-\xi_{\{i,i_3\}}(g^{-S\backslash\{i,i_2,i_3\}})-\xi_{\{i,i_2\}}(g^{-S\backslash\{i,i_2\}}) \\
&= ... \\
&= \xi_i(g)-\sum_{k=2}^{s}\xi_{\{i,i_k\}}(g^{-S\backslash\{i,i_2...,i_k\}}).
\end{aligned}$$

Reiterating this process for $\xi_{\{i,i_k\}}(g^{-S\backslash\{i,i_2...,i_k\}})$, we get, for all $2\leq k\leq s-1$:

$$\xi_{\{i,i_k\}}(g^{-S\backslash\{i,i_2...,i_k\}})=\xi_{\{i,i_k\}}(g)-\sum_{l=k+1}^{s}\xi_{\{i,i_k,i_l\}}(g^{-S\backslash\{i,i_2...,i_l\}}),$$

and, more generally, for all $2\leq k_1<k_2<...<k_q\leq s-1$, we have:

$$\xi_{\{i,i_{k_1},i_{k_2},...,i_{k_q}\}}(g^{-S\backslash\{i,i_2...,i_{k_q}\}})=\xi_{\{i,i_{k_1},i_{k_2},...,i_{k_q}\}}(g)-\sum_{l=k_q+1}^{s}\xi_{\{i,i_{k_1},...,i_{k_q},i_l\}}(g^{-S\backslash\{i,i_2...,i_l\}}).$$

Gathering all expressions together gives:

$$\begin{aligned}\xi_i(g^{-S\setminus\{i\}}) &= \xi_i(g) - \sum_{j\in S\setminus\{i\}} \xi_{\{i,j\}}(g) + \sum_{\{j,k\}\subseteq S\setminus\{i\}} \xi_{\{i,j,k\}}(g) - ... + (-1)^{|S|}\xi_S(g) \\ &= \sum_{\Omega\subseteq S\setminus\{i\}} (-1)^{|\Omega|}\xi_{\Omega\cup\{i\}}(g,\phi).\end{aligned}$$

Plugging back into (13), we finally obtain:

$$\begin{aligned}\beta(g^{-S}) &= \beta(g,\phi) - \sum_{\Omega\subseteq S\setminus\{i\}} (-1)^{|\Omega|+1}\xi_\Omega(g,\phi) - \sum_{\Omega\subseteq S\setminus\{i\}} (-1)^{|\Omega|}\xi_{\Omega\cup\{i\}}(g,\phi) \\ &= \beta(g,\phi) - \sum_{\Omega\subseteq S} (-1)^{|\Omega|+1}\xi_\Omega(g,\phi),\end{aligned}$$

which concludes the proof. ∎

Proof of Proposition 8. From Proposition 1 and (11) with $\widetilde{\phi} = \min\{\widehat{\phi}(\mathbf{c}) \mid \mathbf{c} \in \{0,1\}^n, \mathbf{c} \neq \mathbf{0}\}$. ∎