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Bidding with Coalitional Externalities: A strategic Approach to partition function form Games

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Abstract

This paper provides a strategic approach to the model of partition function form games that is used to analyze coalitional externalities. Two solution concepts are implemented: the Shapley value defined by Pham Do and Norde (2007) and the consensus value by Ju (2007). The building block of the approach is the bidding mechanism introduced in Pérez-Castrillo and Wettstein (2001) and generalized in Ju and Wettstein (2006). Hence, it presents a consistent non-cooperative benchmark to study and compare cooperative solution concepts in various situations.

JEL classification codes: C71; C72; D62.
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1 Introduction

The economic environment featured by coalitional externalities has been effectively modelled by the game theoretic framework of partition function form games proposed by Thrall and Lucas (1963). From the normative point of view, values for such games are studied by, among others, Myerson (1977), Bolger (1989), Feldman (1994), Maskin (2003), de Clippel and Serrano (2005), Macho-Stadler, et. al. (2007), Pham Do and Norde (2007), and Ju (2007). Due to the more complicated structure of partition function form games compared to the standard TU (transferable utility) games, it is generally conceived that to establish the non-cooperative foundations[1] for the associated solution concepts is not straightforward. Consequently, only few work, e.g. Maskin (2003) and Macho-Stadler, et. al. (2006), has addressed this issue.

This paper aims to make further investigation in the direction by implementing the values proposed by Pham Do and Norde (2007) and Ju (2007). The value proposed by Pham Do and Norde (2007) is a direct extension of the Shapley value (Shapley (1953)) to partition function form games. Although this solution concept de facto ignores externalities by construction, it does possess important properties which reflect a certain standpoint over externalities. For more discussion of this value, we refer to Fujinaka (2004) and de Clippel and Serrano (2005). Moreover, here we like to note an interesting point: contrary to the concept's definition that ignores externalities, the underlying non-cooperative mechanism introduced in this paper suggests that it does well consider externalities.

Ju (2007) introduced the consensus value for partition function form games based on a similar idea to define the value (cf. Ju, et. al. (2007)) for TU games. The underlying procedure to construct the value and the axiomatic characterizations show that it well balances the tradeoff between coalitional effects and externality effects in the context. Another desirable feature of the consensus value is that it satisfies individual rationality under a superadditivity condition, whereas the other values do not satisfy.

In this paper, by extending the generalized bidding approach proposed by Ju and Wettstein (2006) to partition function form games, we obtain mechanisms to implement the values proposed by Pham Do and Norde (2007) and Ju (2007), which discovers the interesting strategic features behind them. This approach does not use the structure of any specific value to generate a specific mechanism tailored for it, but, through the bidding, allows players to consider the payoffs and externalities to all possible sub-coalitions. The emergence of a solution concept, not directly related to the mechanism, serves to highlight

[1]Here we use the phrase of non-cooperative "foundations" just for convenience. However, we agree with Serrano (2005) that the normative properties of a solution concept fits better the notion of a "foundation". Serrano (2005) provides an excellent survey for the work of implementations of cooperative solutions.

intriguing features of the solution concept. The consensus value, for example, emerges as equilibrium outcome when players compete for the right to make a second offer rather than arbitrarily assigning it to a particular player.

The findings obtained in the paper not only complements the results obtained in Maskin (2003) and Macho-Stadler, et. al. (2006) but also suggests a unified approach to analyze such games. The design of a single basic mechanism to implement several solution concepts can help to make direct and critical comparison among them and highlight the underlying different non-cooperative rationales. We further show that the approach can serve as a toolkit for analyzing partition function form games, both in implementation itself and in the direction of looking for new solution concepts.

The next section presents the environment and the values to be implemented. In section 3, we formally describe the bidding mechanisms and show that bearing the same bidding stage in all mechanisms, different protocols of renegotiation result in completely different value concepts as equilibrium outcomes. The final section provides concluding remarks.

2 Partition function form games and two values

We now formally present the model of partition function form games. Let N be the set of players. A coalition S is a subset of N. A *partition* κ of N, a so-called *coalition structure*, is a set of mutually disjoint coalitions, $\kappa = \{S_1, ..., S_m\}$, so that their union is N. Let $\mathbb{P}(N)$ denote the set of all partitions of N. For any coalition $S \subseteq N$, $\mathbb{P}(S)$ denotes the set of all partitions of S. A typical element of $\mathbb{P}(S)$ is denoted by κ_S. Note that two partitions will be considered equal if they differ only by the insertion or deletion of $\emptyset$. That is, $\{\{1,2\},\{3\}\} = \{\{1,2\},\{3\},\emptyset\}$. A pair (S,κ) consisting of a coalition S and a partition $\kappa \in \mathbb{P}(N)$ to which S belongs is called an *embedded coalition*, and is nontrivial if $S \neq \emptyset$. Let $\mathbb{E}(N)$ denote the set of embedded coalitions, i.e.

$$\mathbb{E}(N) = \left\{(S,\kappa) \in 2^N \times \mathbb{P}(N) | S \in \kappa\right\}.$$

We denote by (N,w) a game in partition function form (or a *partition function form game*) where $w : \mathbb{E}(N) \longrightarrow \mathbb{R}$ is called a *partition function* that assigns a real value, $w(S,\kappa)$, to each embedded coalition (S,κ). The value $w(S,\kappa)$ represents the payoff of coalition S, given the coalition structure κ forms. By convention, $w(\emptyset,\kappa) = 0$ for all $\kappa \in \mathbb{P}(N)$. The set of partition function form games with player set N is denoted by PG^N.

For a given partition $\kappa = \{S_1, ..., S_m\}$ and a partition function w, let $\bar{w}(S_1, ..., S_m)$ denote the m-vector $(w(S_i,\kappa))_{i=1}^m$. For any $S \subseteq N$ we denote by $[S]$ the partition of S which consists of *singleton coalitions* only, $[S] = \{\{j\}|j \in S\}$, and by $\{S\}$ the partition of S consisting of the coalition S only.

A solution concept on PG^N is a function f, which associates with each game (N, w) in PG^N a vector $f(N, w)$ of individual payoffs in $\mathbb{R}^N$, i.e. $f(N, w) = (f_i(N, w))_{i \in N} \in \mathbb{R}^N$.

Given a partition function form game (N, w) and a subset $S \subseteq N$, we define the subgame $(S, w|_S)$ by assigning the value $w|_S(T, \kappa_S) \equiv w(T, \kappa_S \cup [N \backslash S])$ for all $(T, \kappa_S) \in \mathbb{E}(S)$.

We first recall the Shapley value defined by Pham Do and Norde (2007). Let $\Pi(N)$ be the set of all bijections $\sigma : \{1, ..., |N|\} \longrightarrow N$. For a given $\sigma \in \Pi(N)$ and $k \in \{1, ..., |N|\}$, we define the partition κ_k^σ associated with σ and k, by $\kappa_k^\sigma = \{S_k^\sigma\} \cup [N \backslash S_k^\sigma]$ where $S_k^\sigma := \{\sigma(1), ..., \sigma(k)\}$, and $\kappa_0^\sigma = [N]$. So, in κ_k^σ the coalition S_k^σ has already formed, whereas all other players still form singleton coalitions. For a game $w \in PG^N$, define the marginal vector $m^\sigma(w)$ as the vector in $\mathbb{R}^N$ by $m^\sigma_{\sigma(k)}(w) = w(S_k^\sigma, \kappa_k^\sigma) - w(S_{k-1}^\sigma, \kappa_{k-1}^\sigma)$ for all $\sigma \in \Pi(N)$ and $k \in \{1, ..., |N|\}$. The *Shapley value* (Pham Do and Norde (2007)) $\phi(w)$ of the partition function form game (N, w) is defined as the average, over the set $\Pi(N)$ of all bijections, of the marginal vectors, i.e.

$$\phi(w) = \frac{1}{|N|!} \sum_{\sigma \in \Pi(N)} m^\sigma(w).$$

It is the unique value satisfying efficiency, additivity, symmetry and the null player property.

Ju (2007) introduces the consensus value for partition function form games by taking a bilateral perspective and considering both coalitional effects and externality effects when sharing the gains of cooperation. The consensus value is the unique solution that satisfies efficiency, complete symmetry, additivity and the quasi-null player property. It is shown that the consensus value for partition function form games γ equals the average of the Shapley value (Pham Do and Norde (2007)) and the expected stand-alone value. That is, $\gamma(w) = \frac{1}{2}\phi(w) + \frac{1}{2}e(w)$, where $e(w)$ denotes the expected stand-alone value of a partition function form game w and defined by

$$\begin{aligned} e_i(w) = \quad & \frac{w(N, \{N\})}{|N|} \\ + & \sum_{S \subseteq N \backslash \{i\} : S \neq \emptyset} \frac{|S|!(|N| - |S| - 1)!}{|N|!} w(\{i\}, \{S\} \cup [N \backslash (S \cup \{i\})] \cup \{\{i\}\}) \\ - & \sum_{j \in N \backslash \{i\}} \sum_{S \subseteq N \backslash \{i,j\}} \frac{|S|!(|N| - |S| - 2)!}{|N|!} w(\{j\}, [N \backslash (S \cup \{i\})] \cup \{S \cup \{i\}\}). \end{aligned}$$

for all $i \in N$. The expected stand-alone value takes players' stand-alone situations as the only decisive input to determine their final payoffs. Hence, it purely measures the externality effects in a partition function form game, compared to the coalitional effects by the Shapley value. For more details of these values, we refer to Ju (2007).

3 The bidding mechanisms

In this section, we construct bidding mechanisms that implement the above two cooperative solutions for partition function form games. These mechanisms provide a convenient benchmark to evaluate and compare these values from a non-cooperative perspective.

The basic *bidding mechanism* can be described informally as follows: At stage 1 the players bid to choose a proposer. Each player bids by submitting an $(n-1)$-tuple of numbers (positive or negative), one number for each player (excluding herself). The player for whom the net bid (the difference between the sum of bids made by the player and the sum of bids the other players made to her, measuring the player's willingness to become the proposer) is the highest, is chosen as the proposer. Before moving to stage 2, the proposer pays to each player the bid she made. As a reward to the chosen proposer for her effort (represented by her net bid), she is granted with the right to make a scheme how to split the total payoffs $w(N, \{N\})$ among all the players at the next stage.

At stage 2 the proposer offers a vector of payments to all other players in exchange for joining her to form the grand coalition. The offer is accepted if all the other players agree. In case of acceptance the grand coalition indeed forms and the proposer receives $w(N, \{N\})$ out of which she pays out the offers made. In case of rejection the proposer "waits" while all the other players go again through the same game.

What are the possible consequences following this rejection? In general, there can be two different scenarios. One is that all the remaining players fail to reach any agreement among themselves again. Then it is not difficult to imagine a natural outcome: the hope of forming the grand coalition collapses and the initial proposer will be indeed left alone. The other scenario might be that the remaining players do agree on a proposal within them, which means that a coalition of all players apart from the initial proposer is formed. In this case, the option of "re-entering" the game for the initial proposer becomes realistic. Since now it is a two-party issue, given the potential benefit from cooperation, it is very reasonable for them to come back to the table and negotiate again. The following stages will be associated with this renegotiation. That is, in these additional stages the first proposer (in fact, the rejected proposer) and the proposer chosen among the remaining players (when an agreement is reached within themselves) bid and accept further offers (note that these stages are also present in the game played by any remaining group of players).

The first variant implementing the Shapley value has the first proposer (denoted for simplicity by a) make an offer to the proposer chosen among the remaining players (denoted for simplicity by b). The offer is for a to form the grand coalition rather than b. If the offer is accepted the grand coalition forms, a receives $w(N, \{N\})$ and pays the offer, b receives the offer from a and pays all the commitments made by him, and all the other players receive what they were promised. In this variant a retains the right to make offers.

The second variant implementing the expected stand-alone value has b make an offer to a. If the offer is accepted the grand coalition forms, a receives the offer, b receives $w(N, \{N\})$ and pays the offer to a as well as what he owes to the remaining players. In this variant a loses the right to make offers.

In the third variant implementing the consensus value a and b bid for the right to make an offer. If a wins the game proceeds as in the first variant and if b wins the second variant goes into effect.

Below we formally describe the bidding mechanisms, which will explicitly explain how these bargaining protocols deal with coalitional externalities.

Mechanism A. If there is only one player $\{i\}$, she simply receives $w(i, \{i\})$. When there are two or more players, the mechanism is defined recursively. Given the rules of the mechanism for at most $n-1$ players, the mechanism for $N = \{1, \ldots, n\}$ proceeds in five stages.

Stage 1: Each player $i \in N$ makes $n-1$ bids $b_j^i \in \mathbb{R}$ with $j \neq i$.
For each $i \in N$, define the *net bid* to player i by $B^i = \sum_{j \neq i} b_j^i - \sum_{j \neq i} b_i^j$. Let $i^* = \text{argmax}_i(B^i)$ where an arbitrary tie-breaking rule is used in case of a non-unique maximizer. Once the winner i^* has been chosen, player i^* pays every player $j \in N \backslash \{i^*\}$, $b_j^{i^*}$.

Stage 2: Player i^* makes a vector of offers $x_j^{i^*} \in \mathbb{R}$ to every player $j \in N \backslash \{i^*\}$.

Stage 3: The players other than i^*, sequentially, either accept or reject the offer. If at least one player rejects it, then the offer is rejected. Otherwise, the offer is accepted.
If the offer is accepted, which means that all players agree with the proposer on the scheme of sharing $w(N, \{N\})$, then each player $j \in N \backslash \{i^*\}$ receives $x_j^{i^*}$ at this stage, and player i^* receives $w(N, \{N\}) - \sum_{j \neq i^*} x_j^{i^*}$. Hence, the final payoff to player $j \neq i^*$ is $x_j^{i^*} + b_j^{i^*}$ while player i^* receives $w(N, \{N\}) - \sum_{j \neq i^*} x_j^{i^*} - \sum_{j \neq i^*} b_j^{i^*}$.
If the offer made by the proposer i^* is rejected, all players other than i^* proceed to play a similar game with one player less, i.e., with the set of players $N \backslash \{i^*\}$. By the same bidding stage, the newly chosen proposer among $N \backslash \{i^*\}$, denoted j^*, will make an offer to players in $N \backslash \{i^*, j^*\}$. If accepted, j^* pays the offer and coalition $N \backslash \{i^*\}$ forms. However, the payoff of j^* relies on the final coalition structure of all players N, which is further dependent upon how the renegotiation between i^* and j^* proceeds at stages 4 and 5 and whether or not they can make an agreement.

If the offer made by j^* to $N\backslash\{i^*, j^*\}$ is rejected, there is one more chance for $N\backslash\{i^*, j^*\}$ and j^* to be reunited into $N\backslash\{i^*\}$ by renegotiation following the same rule specified below at stages 4 and 5 (but with the player set $N\backslash\{i^*\}$). Then, as long as coalition $N\backslash\{i^*\}$ emerges, either due to the immediate acceptance of the offer made by j^* or by agreement on renegotiation between $N\backslash\{i^*, j^*\}$ and j^*, the game moves to stage 4 and the renegotiation between i^* and j^* (who is also representing $N\backslash\{i^*\}$) takes place. In case no agreement is reached by $N\backslash\{i^*\}$ and thereby coalition $N\backslash\{i^*\}$ does not emerge, player i^* loses the option of renegotiating with $N\backslash\{i^*\}$ and is indeed left alone and gets a stand-alone payoff that depends on the coalition structure of $N\backslash\{i^*\}$ at this stage.

Stage 4: Player i^* makes an offer $\widetilde{x}_{j^*}^{i^*}$ in $\mathbb{R}$, to the proposer j^* chosen among the set of players $N\backslash\{i^*\}$. (The offer is to let i^* form the grand coalition instead of player j^*.)

Stage 5: Player j^* accepts or rejects the offer. If the offer is accepted then at this stage player j^* receives $\widetilde{x}_{j^*}^{i^*}$ minus the bids and offer he made to the players in $N\backslash\{i^*, j^*\}$, while player i^* receives $w(N, \{N\}) - \widetilde{x}_{j^*}^{i^*}$. Hence, the final payoff to player j^* is $\widetilde{x}_{j^*}^{i^*} + b_{j^*}^{i^*}$ minus the bids and offer he made to the players in $N\backslash\{i^*, j^*\}$, and player i^* receives $w(N, \{N\}) - \widetilde{x}_{j^*}^{i^*} - \sum_{j \neq i^*} b_j^{i^*}$, whereas each player $k \in N\backslash\{i^*, j^*\}$ receives the payoff of the outcome in the subgame played by $N\backslash\{i^*\}$ plus the bid $b_k^{i^*}$. If the offer is rejected, it implies that two different coalitions form, i.e., coalition $N\backslash\{i^*\}$ and the singleton coalition $\{i^*\}$ so that the final coalition structure of N is $\{N\backslash\{i^*\}\} \cup \{\{i^*\}\}$. Then, each player $k \neq N\backslash\{i^*, j^*\}$ finally receives the payoff resulting from the subgame played by $N\backslash\{i^*\}$ in addition to $b_k^{i^*}$, the final payoff of player j^* is $w(N\backslash\{i^*\}, \{N\backslash\{i^*\}\} \cup \{\{i^*\}\})$ minus the bids and offers that he made to all players in $N\backslash\{i^*, j^*\}$ and plus the bid from i^*, $b_{j^*}^{i^*}$, and player i^* receives $w(\{i^*\}, \{N\backslash\{i^*\}\} \cup \{\{i^*\}\}) - \sum_{j \neq i^*} b_j^{i^*}$.

The following theorem shows that for any partition function form game (N, w) satisfying zero-monotonicity, i.e.,

$$w(S, \{S\} \cup [N\backslash S]) \geq w(S\backslash\{i\}, \{S\backslash\{i\}\} \cup [N\backslash(S\backslash\{i\})]) + w(\{i\}, \{S\backslash\{i\}\} \cup [N\backslash(S\backslash\{i\})])$$

for all $S \subseteq N$ and $i \in S$, the subgame perfect equilibrium (SPE) outcomes of Mechanism A coincide with the payoff vector $\phi(N, w)$ as prescribed by the Shapley value defined by Pham Do and Norde (2007).

Theorem 3.1 *Mechanism A implements the Shapley value defined by Pham Do and Norde (2007) of a zero-monotonic partition function form game* (N, w) *in SPE.*

Proof. Let (N, w) be a zero-monotonic partition function form game. The proof proceeds by induction on the number of players n. It is easy to see that the theorem holds for $n = 1$. We assume that it holds for all $m \leq n-1$ and show that it is satisfied for n.

First we show that the Shapley value is an SPE outcome. We explicitly construct an SPE that yields the Shapley value as an SPE outcome. Consider the following strategies, which the players would follow in any game they participate in (we describe it for the whole set of players, N, but these are also the strategies followed by any player in a subset S that is called upon to play the game, with S replacing N):

At stage 1, each player $i \in N$, announces $b_j^i = \phi_j(N, w) - \phi_j(N\backslash\{i\}, w|_{N\backslash\{i\}})$ for every $j \in N\backslash\{i\}$.

At stage 2, a proposer, player i^*, offers $x_j^{i^*} = \phi_j(N\backslash\{i^*\}, w|_{N\backslash\{i^*\}})$ to every $j \in N\backslash\{i^*\}$.

At stage 3, any player $j \in N\backslash\{i^*\}$ accepts any offer which is greater than or equal to $\phi_j(N\backslash\{i^*\}, w|_{N\backslash\{i^*\}})$ and rejects any offer strictly less than $\phi_j(N\backslash\{i^*\}, w|_{N\backslash\{i^*\}})$.

At stage 4, player i^* makes an offer $\widetilde{x}_{j^*}^{i^*} = w(N\backslash\{i^*\}, \{N\backslash\{i^*\}\} \cup \{\{i^*\}\})$ to any selected proposer $j^* \in N\backslash\{i^*\}$.

At stage 5, player j^*, the proposer of the set of players $N\backslash\{i^*\}$, accepts any offer greater than or equal to $w(N\backslash\{i^*\}, \{N\backslash\{i^*\}\} \cup \{\{i^*\}\})$ and rejects any offer strictly less than it.

Clearly these strategies yield the Shapley value for any player who is not the proposer, since the game ends at stage 3 and $b_j^{i^*} + x_j^{i^*} = \phi_j(N, w)$, for all $j \neq i^*$. Moreover, given that following the strategies the offer is accepted by all players, the proposer also obtains her Shapley value.
Note that all net bids equal zero by the balanced contribution property for the Shapley value (Myerson (1980)).
To show that the previous strategies constitute an SPE, note first that the strategies at stages 2, 3, 4, and 5 are best responses: In case of rejection at stage 3 proposer i^* can obtain $w(N, \{N\}) - w(N\backslash\{i^*\}, \{N\backslash\{i^*\}\} \cup \{\{i^*\}\})$ in the end (it pays her to make an offer that is accepted at stage 4, by zero-monotonicity), and all other players play the bidding mechanism with player set $N\backslash\{i^*\}$ and payoff $w(N\backslash\{i^*\}, \{N\backslash\{i^*\}\} \cup \{\{i^*\}\})$. By the induction hypothesis, we have the Shapley value as the outcome of this game. That is, each player $j \in N\backslash\{i^*\}$ gets $\phi_j(N\backslash\{i^*\}, w|_{N\backslash\{i^*\}})$. Consider now the strategies at stage 1. If player i^* increases her total bid, then she will be chosen as the proposer with certainty, but her payoff will decrease. If she decreases her total bid another player will propose and player i^*'s payoff would still equal her Shapley value. Finally, any change in her bids that

leaves the total bid constant will influence the identity of the proposer but will not affect player i^*'s payoff.

The proof that any SPE yields the Shapley value proceeds in the same line as in the proof of Theorem 3.1 in Ju and Wettstein (2006) and therefore skipped. ■

The key feature of Mechanism A in implementing the Shapley value is that it allows the proposer chosen from the first bidding stage to have the power of making another offer at stage 4 in case she has been rejected at stage 3.[2] One might argue that the right to make a second offer should be awarded to the new proposer who is chosen from the remaining players rather than the original proposer whose offer has been rejected. Such an argument would lead to a new mechanism, which implements the expected stand-alone value.

Mechanism B. Stages $1, 2$ and 3 are the same as in Mechanism A. Note that in case of rejection at stage 3, the game played by $N\backslash\{i^*\}$ will follow, when renegotiations within $N\backslash\{i^*\}$ are called for, the rules specified at stages 4 and 5 of the current mechanism.

Stage 4: Player j^*, the proposer chosen among the set of players $N\backslash\{i^*\}$ makes an offer $\widetilde{x}_{i^*}^{j^*}$ in $\mathbb{R}$, to player i^*. (The offer is to pay i^* this amount for joining in to form the grand coalition).

Stage 5: Player i^* accepts or rejects the offer. If the offer is accepted then at this stage each player $k \in N\backslash\{i^*, j^*\}$ receives the payoff of the outcome in the subgame played by $N\backslash\{i^*\}$, player j^* receives $w(N, \{N\}) - \widetilde{x}_{i^*}^{j^*}$ minus the bids and offer he made to the players in $N\backslash\{i^*, j^*\}$, and player i^* receives $\widetilde{x}_{i^*}^{j^*}$. Hence, the final payoff to player $k \in N\backslash\{i^*, j^*\}$ is the payoff of the outcome in the subgame played by $N\backslash\{i^*\}$ plus $b_k^{i^*}$; player j^* finally receives $w(N, \{N\}) - \widetilde{x}_{i^*}^{j^*} + b_{j^*}^{i^*}$ minus the bids and offer he made to the players in $N\backslash\{i^*, j^*\}$, and player i^* finally receives $\widetilde{x}_{i^*}^{j^*} - \sum_{j \neq i^*} b_j^{i^*}$. If the offer is rejected, we then have the final coalition structure of N as $\{N\backslash\{i^*\}\} \cup \{\{i^*\}\}$. Then, each player $k \neq N\backslash\{i^*, j^*\}$ finally receives the payoff resulting from the subgame played by $N\backslash\{i^*\}$ in addition to $b_k^{i^*}$, the final payoff of player j^* is $w(N\backslash\{i^*\}, \{N\backslash\{i^*\}\} \cup \{\{i^*\}\})$ minus the bids and offers that he made to all players in $N\backslash\{i^*, j^*\}$ and plus the bid from i^*, $b_{j^*}^{i^*}$, and player i^* finally receives $w(\{i^*\}, \{N\backslash\{i^*\}\} \cup \{\{i^*\}\}) - \sum_{j \neq i^*} b_j^{i^*}$.

[2] Hence, one can readily think of an alternative mechanism with no renegotiation that still implements the Shapley value. Such a mechanism will only involve stages 1, 2, and 3 in which a rejected proposer has no option of renegotiation and is left alone with payoff dependent upon the coalition structure of the remaining players.

Before stating the main result about Mechanism B, we show the following lemma. Let us first define the subgame $(N\backslash\{i\}, w^{-i})$ of (N, w) by

$$w^{-i}(N\backslash\{i\}, \{N\backslash\{i\}\}) = w(N, \{N\}) - w(\{i\}, \{N\backslash\{i\}\} \cup \{\{i\}\})$$

and

$$w^{-i}(S, \kappa_{N\backslash\{i\}}) = w(S, \kappa_{N\backslash\{i\}} \cup \{\{i\}\})$$

for all $(S, \kappa_{N\backslash\{i\}}) \in \mathbb{E}(N\backslash\{i\})\backslash(N\backslash\{i\}, N\backslash\{i\})$.

Lemma 3.2 *For any game $w \in PG^N$ we have*

$$\sum_{j\in N\backslash\{i\}} \big(e_j(N, w) - e_j(N\backslash\{i\}, w^{-i})\big) - \sum_{j\in N\backslash\{i\}} \big(e_i(N, w) - e_i(N\backslash\{j\}, w^{-j})\big) = 0$$

for all $i, j \in N$.

Proof.
By the definition of the expected stand-alone value, it suffices to show that

$$-|N|e_i(N, w) + w(\{i\}, \{N\backslash\{i\}\} \cup \{\{i\}\}) + \sum_{j\in N\backslash\{i\}} e_i(N\backslash\{j\}, w^{-j}) = 0$$

for all $i, j \in N$ and $i \neq j$. Obviously,

$$\begin{aligned} |N|e_i(N, w) = \quad & w(N, \{N\}) \\ + & \sum_{S\subseteq N\backslash\{i\}:S\neq\emptyset} \frac{|S|!(|N| - |S| - 1)!}{(|N| - 1)!} w(\{i\}, \{S\} \cup [N\backslash(S \cup \{i\})] \cup \{\{i\}\}) \\ - & \sum_{k\in N\backslash\{i\}} \sum_{S\subseteq N\backslash\{i,k\}} \frac{|S|!(|N| - |S| - 2)!}{(|N| - 1)!} w(\{k\}, [N\backslash(S \cup \{i\})] \cup \{S \cup \{i\}\}). \end{aligned}$$

and

$$\begin{aligned} & \sum_{j\in N\backslash\{i\}} e_i(N\backslash\{j\}, w^{-j}) \\ = \quad & w(N, \{N\}) - \sum_{j\in N\backslash\{i\}} \frac{1}{|N| - 1} w(\{j\}, \{N\backslash\{j\}\} \cup \{\{j\}\}) \\ + & \sum_{j\in N\backslash\{i\}} \sum_{S\subseteq N\backslash\{i,j\}:S\neq\emptyset} \frac{|S|!(|N| - |S| - 2)!}{(|N| - 1)!} w(\{i\}, \{S\} \cup [N\backslash(S \cup \{i\})] \cup \{\{i\}\} \cup \{\{j\}\}) \\ - & \sum_{j\in N\backslash\{i\}} \sum_{k\in N\backslash\{i,j\}} \sum_{S\subseteq N\backslash\{i,j,k\}} \frac{|S|!(|N| - |S| - 3)!}{(|N| - 1)!} w(\{k\}, [N\backslash(S \cup \{i\})] \cup \{S \cup \{i\}\} \cup \{\{j\}\}). \end{aligned}$$

Moreover, we know that

$$
\begin{aligned}
& \sum_{S\subseteq N\backslash\{i\}:S\neq\emptyset} \frac{|S|!(|N|-|S|-1)!}{(|N|-1)!} w(\{i\},\{S\}\cup[N\backslash(S\cup\{i\})]\cup\{\{i\}\}) \\
= & \sum_{S=N\backslash\{i\}:S\neq\emptyset} \frac{|S|!(|N|-|S|-1)!}{(|N|-1)!} w(\{i\},\{S\}\cup[N\backslash(S\cup\{i\})]\cup\{\{i\}\}) \\
+ & \sum_{S\subsetneqq N\backslash\{i\}:S\neq\emptyset} \frac{|S|!(|N|-|S|-1)!}{(|N|-1)!} w(\{i\},\{S\}\cup[N\backslash(S\cup\{i\})]\cup\{\{i\}\}) \\
= & \; w(\{i\},\{N\backslash\{i\}\}\cup\{\{i\}\}) \\
+ & \sum_{j\in N\backslash\{i\}} \sum_{S\subseteq N\backslash\{i,j\}:S\neq\emptyset} \frac{|S|!(|N|-|S|-2)!}{(|N|-1)!} w(\{i\},\{S\}\cup[N\backslash(S\cup\{i\})]\cup\{\{i\}\}\cup\{\{j\}\})
\end{aligned}
$$

and similarly,

$$
\begin{aligned}
& \sum_{j\in N\backslash\{i\}} \sum_{k\in N\backslash\{i,j\}} \sum_{S\subseteq N\backslash\{i,j,k\}} \frac{|S|!(|N|-|S|-3)!}{(|N|-1)!} w(\{k\},[N\backslash(S\cup\{i\})]\cup\{S\cup\{i\}\}\cup\{\{j\}\}) \\
= & \sum_{k\in N\backslash\{i\}} \sum_{S\subsetneqq N\backslash\{i,k\}} \frac{|S|!(|N|-|S|-2)!}{(|N|-1)!} w(\{k\},[N\backslash(S\cup\{i\})]\cup\{S\cup\{i\}\}).
\end{aligned}
$$

What remain is clear because

$$
\begin{aligned}
& \sum_{S=N\backslash\{i,j\}} \frac{|S|!(|N|-|S|-2)!}{(|N|-1)!} w(\{j\},[N\backslash(S\cup\{i\})]\cup\{S\cup\{i\}\}) \\
= & \frac{1}{|N|-1} w(\{j\},\{N\backslash\{j\}\}\cup\{\{j\}\}).
\end{aligned}
$$

■

Theorem 3.3 *Mechanism B implements the expected stand-alone value of a zero-monotonic partition function form game* (N,w) *in SPE.*

Proof. The proof is analogous to that of Theorem 3.1. The differences are in the construction of the SPE strategies and in showing that in any SPE, the final payment received by each of the players coincides with each player's expected stand-alone value. Hence, we first explicitly construct an SPE that yields the expected stand-alone value as an SPE outcome.

To constuct an SPE, consider the following strategies.

At stage 1, each player $i \in N$, announces $b^i_j = e_j(N,w) - e_j(N\backslash\{i\}, w^{-i})$, for every $j \in N\backslash\{i\}$.

At stage 2, a proposer, player i^*, offers $x_j^{i^*} = e_j(N\backslash\{i^*\}, w^{-i^*})$ to every $j \in N\backslash\{i^*\}$.

At stage 3, any player $j \in N\backslash\{i^*\}$ accepts any offer which is greater than or equal to $e_j(N\backslash\{i^*\}, w^{-i^*})$ and rejects any offer strictly less than $e_j(N\backslash\{i^*\}, w^{-i^*})$.

At stage 4, a proposer within $N\backslash\{i^*\}$, player j^* makes an offer $\widetilde{x}_{i^*}^{j^*} = w(\{i^*\}, \{N\backslash\{i^*\}\} \cup \{\{i^*\}\})$ to i^*.

At stage 5, player i^*, the "waiting" proposer for the set of players N, accepts any offer greater than or equal to $w(\{i^*\}, \{N\backslash\{i^*\}\} \cup \{\{i^*\}\})$ and rejects any offer strictly less than it.

One can readily verify that these strategies yield the equal surplus value for any player and constitute an SPE.

Next we show that in any SPE the final payment received by each of the players coincides with each player's expected stand-alone value. We note that if i is the proposer, her final payoff will be $w(N, \{N\}) - (w(N, \{N\}) - w(\{i^*\}, \{N\backslash\{i^*\}\} \cup \{\{i^*\}\})) - \sum_{j \neq i} b_j^i$, whereas if $j \neq i$ is the proposer, i will get final payoff $e_i(N\backslash\{j\}, w^{-j}) + b_i^j$. Hence the sum of the payoffs to player i over all possible choices is (note that all net bids are zero)

$$\begin{aligned} & w(N, \{N\}) - (w(N, \{N\}) - w(\{i^*\}, \{N\backslash\{i^*\}\} \cup \{\{i^*\}\})) - \sum_{j \neq i} b_j^i \\ + & \sum_{j \neq i} \left(e_i(N\backslash\{j\}, w^{-j}) + b_i^j\right) \\ = \quad & w(\{i^*\}, \{N\backslash\{i^*\}\} \cup \{\{i^*\}\}) + \sum_{j \neq i} e_i(N\backslash\{j\}, w^{-j}), \end{aligned}$$

which, by Lemma 3.2, equals $n \cdot e_i(N, v)$. Since the payoffs are the same regardless of who is the proposer we see that the payoff of each player in any equilibrium must coincide with the expected stand-alone value. ■

The above mechanisms A and B take extreme and contrastive treatments in case an offer is rejected, which give *a priori* full power to either the initial proposer or the proposer chosen from the set of remaining players to make a second offer. A less biased option would be giving equal power to the two proposers to make a second offer. That is, let the two compete (by bidding) for the role of being the proposer to make a further offer when they engage in renegotiation. This mechanism as formally described below implements the consensus value.

Mechanism C. The rules of stages 1, 2 and 3 are the same as before. Of course, in case of rejection at stage 3, the game played by $N\backslash\{i^*\}$ will follow, when renegotiations within $N\backslash\{i^*\}$ are called for, the rules specified at stages 4 and 5 of this mechanism.

Stage 4: Player i^*, the proposer chosen among N, and player j^*, the proposer chosen among the set of players $N\backslash\{i^*\}$, bid for the right to take the role of the proposer (the game played, in fact, coincides with the stage 1 game with $n = 2$). Both i^* and j^* simultaneously submit bids $\widetilde{b}^{i^*}_{j^*}$ and $\widetilde{b}^{j^*}_{i^*}$ in $\mathbb{R}$. The player with the larger net bid pays the bid to the other player and assumes the role of the proposer. In case of identical bids the proposer is chosen randomly.

Stage 5: Depending on whether the proposer is i^* or j^*, the game proceeds as in Mechanism C1 (when i^* is the proposer) or Mechanism C2 (when j^* is the proposer). The payoffs are adjusted to take into account the bidding at stage 4.

Lemma 3.4 *For any game $w \in PG^N$ we have*

$$\begin{aligned} & |N|e_i(N,w) \\ = \quad & w(\{i\},\{N\backslash\{i\}\}\cup\{\{i\}\}) \\ + \quad & \sum_{j\in N\backslash\{i\}} \frac{w(N,\{N\}) - w(N\backslash\{j\},\{N\backslash\{j\}\}\cup\{\{j\}\}) - w(\{j\},\{N\backslash\{j\}\}\cup\{\{j\}\})}{|N|-1} \\ + \quad & \sum_{j\in N\backslash\{i\}} e_i(N\backslash\{j\}, w|_{N\backslash\{j\}}) \end{aligned}$$

for all $i \in N$.

Proof. The proof can be constructed along the same line as that for Lemma 3.2. ■

Theorem 3.5 *Mechanism C implements the consensus value of a zero-monotonic partition function form game (N,w) in SPE.*

Proof. The proof is again similar to that of Theorem 3.1. The differences are once more in the construction of the SPE strategies and in claiming that payoffs must coincide with the consensus value. To explicitly construct an SPE that yields the consensus value, consider the following strategies.

At stage 1, each player $i \in N$ announces $b^i_j = \gamma_j(N,v) - \gamma_j(N\backslash\{i\},\widehat{w}^{-i})$,[3] for every $j \in N\backslash\{i\}$.

At stage 2, a proposer, player i^*, offers $x^{i^*}_j = \gamma_j(N\backslash\{i^*\},\widehat{w}^{-i})$ to every $j \in N\backslash\{i^*\}$.

[3]The game $(N\backslash\{i\},\widehat{w}^{-i})$ is formally defined by $\widehat{w}^{-i}(N\backslash\{i\},\{N\backslash\{i\}\}) = w(N\backslash\{i\},\{N\backslash\{i\}\}\cup\{\{i\}\}) + \frac{w(N,\{N\})-w(N\backslash\{i\},\{N\backslash\{i\}\}\cup\{\{i\}\})-w(\{i\},\{N\backslash\{i\}\}\cup\{\{i\}\})}{2}$ and $\widehat{w}^{-i}(S,\kappa_{N\backslash\{i\}}) = w(S,\kappa_{N\backslash\{i\}}\cup\{\{i\}\})$, for all $(S,\kappa_{N\backslash\{i\}}) \in \mathbb{E}(N\backslash\{i\})\backslash(N\backslash\{i\},N\backslash\{i\})$.

At stage 3, any player $j \in N\backslash\{i^*\}$ accepts any offer which is greater than or equal to $\gamma_j(N\backslash\{i^*\}, \widehat{w}^{-i^*})$ and rejects any offer strictly less than $\gamma_j(N\backslash\{i^*\}, \widehat{w}^{-i^*})$.

At stage 4, player i^* announces

$$\begin{aligned}\widetilde{b}^{i^*}_{j^*} = & \quad w(N\backslash\{i^*\}, \{N\backslash\{i^*\}\} \cup \{\{i^*\}\}) \\ & + \frac{w(N, \{N\}) - w(N\backslash\{i^*\}, \{N\backslash\{i^*\}\} \cup \{\{i^*\}\}) - w(\{i^*\}, \{N\backslash\{i^*\}\} \cup \{\{i^*\}\})}{2} \\ & - w(N\backslash\{i^*\}, \{N\backslash\{i^*\}\} \cup \{\{i^*\}\}) \\ = & \quad \frac{v(N) - w(N\backslash\{i^*\}, \{N\backslash\{i^*\}\} \cup \{\{i^*\}\}) - w(\{i^*\}, \{N\backslash\{i^*\}\} \cup \{\{i^*\}\})}{2}\end{aligned}$$

while player j^* announces

$$\begin{aligned}\widetilde{b}^{j^*}_{i^*} = & \quad w(\{i^*\}, \{N\backslash\{i^*\}\} \cup \{\{i^*\}\}) \\ & + \frac{w(N, \{N\}) - w(\{i^*\}, \{N\backslash\{i^*\}\} \cup \{\{i^*\}\}) - w(N\backslash\{i^*\}, \{N\backslash\{i^*\}\} \cup \{\{i^*\}\})}{2} \\ & - w(\{i^*\}, \{N\backslash\{i^*\}\} \cup \{\{i^*\}\}) \\ = & \quad \frac{v(N) - w(\{i^*\}, \{N\backslash\{i^*\}\} \cup \{\{i^*\}\}) - w(N\backslash\{i^*\}, \{N\backslash\{i^*\}\} \cup \{\{i^*\}\})}{2}.\end{aligned}$$

At stage 5, player i^* makes an offer $\widetilde{x}^{i^*}_{j^*} = w(N\backslash\{i^*\}, \{N\backslash\{i^*\}\} \cup \{\{i^*\}\})$ to j^* and player j^* makes an offer $\widetilde{x}^{j^*}_{i^*} = w(\{i^*\}, \{N\backslash\{i^*\}\} \cup \{\{i^*\}\})$ to i^*. Moreover, i^* accepts any offer greater than or equal to $w(\{i^*\}, \{N\backslash\{i^*\}\} \cup \{\{i^*\}\})$ and rejects any offer strictly less than it. Similarly, j^* accepts any offer greater than or equal to $w(N\backslash\{i^*\}, \{N\backslash\{i^*\}\} \cup \{\{i^*\}\})$ and rejects any offer strictly less than it.

One can readily verify that these strategies yield the consensus value for any player and constitute an SPE.

To show that in any SPE each player's final payoff coincides with her consensus value, we note that if i is the proposer her final payoff is given by

$$\begin{aligned} & w(N, \{N\}) - w(N\backslash\{i\}, \{N\backslash\{i\}\} \cup \{\{i\}\}) \\ - & \frac{w(N, \{N\}) - w(N\backslash\{i\}, \{N\backslash\{i\}\} \cup \{\{i\}\}) - w(\{i\}, \{N\backslash\{i\}\} \cup \{\{i\}\})}{2} - \sum_{j \neq i} b^i_j\end{aligned}$$

whereas if $j \neq i$ is the proposer, the final payoff of i is $\gamma_i(N\backslash\{j\}, \widehat{w}^{-j}) + b^j_i$.

Hence the sum of payoffs to player i over all possible choices of the proposer is (again note that all net bids are zero)

$$
\begin{aligned}
& w(N,\{N\}) - w(N\backslash\{i\},\{N\backslash\{i\}\}\cup\{\{i\}\}) \\
- & \frac{w(N,\{N\}) - w(N\backslash\{i\},\{N\backslash\{i\}\}\cup\{\{i\}\}) - w(\{i\},\{N\backslash\{i\}\}\cup\{\{i\}\})}{2} - \sum_{j\neq i} b^i_j \\
+ & \sum_{j\neq i}\left(\gamma_i(N\backslash\{j\},\widehat{w}^{-j}) + b^j_i\right) \\
= & \frac{w(N,\{N\}) - w(N\backslash\{i\},\{N\backslash\{i\}\}\cup\{\{i\}\}) + w(\{i\},\{N\backslash\{i\}\}\cup\{\{i\}\})}{2} \\
+ & \sum_{j\neq i}\left(\frac{1}{2}\phi_i(N\backslash\{j\},\widehat{w}^{-j}) + \frac{1}{2}e_i(N\backslash\{j\},\widehat{w}^{-j})\right) \\
= & \frac{w(N,\{N\}) - w(N\backslash\{i\},\{N\backslash\{i\}\}\cup\{\{i\}\}) + w(\{i\},\{N\backslash\{i\}\}\cup\{\{i\}\})}{2} \\
+ & \frac{1}{2}\sum_{j\neq i}\left(\phi_i(N\backslash\{j\},w|_{N\backslash\{j\}}) + \frac{\frac{w(N,\{N\})-w(N\backslash\{j\},\{N\backslash\{j\}\}\cup\{\{j\}\})-w(\{j\},\{N\backslash\{j\}\}\cup\{\{j\}\})}{2}}{n-1}\right) \\
+ & \frac{1}{2}\sum_{j\neq i}\left(e_i(N\backslash\{j\},w|_{N\backslash\{j\}}) + \frac{\frac{w(N,\{N\})-w(N\backslash\{j\},\{N\backslash\{j\}\}\cup\{\{j\}\})-w(\{j\},\{N\backslash\{j\}\}\cup\{\{j\}\})}{2}}{n-1}\right) \\
= & \frac{1}{2}\left(w(N,\{N\}) - w(N\backslash\{i\},\{N\backslash\{i\}\}\cup\{\{i\}\}) + \sum_{j\neq i}\phi_i(N\backslash\{j\},w|_{N\backslash\{j\}})\right) \\
+ & \frac{1}{2}w(\{i\},\{N\backslash\{i\}\}\cup\{\{i\}\}) \\
+ & \frac{1}{2}\sum_{j\neq i}\left(\frac{w(N,\{N\}) - w(N\backslash\{j\},\{N\backslash\{j\}\}\cup\{\{j\}\}) - w(\{j\},\{N\backslash\{j\}\}\cup\{\{j\}\})}{n-1}\right) \\
+ & \frac{1}{2}\sum_{j\neq i} e_i(N\backslash\{j\},w|_{N\backslash\{j\}}),
\end{aligned}
$$

which, since $w(N,\{N\})-w(N\backslash\{i\},\{N\backslash\{i\}\}\cup\{\{i\}\})+\sum_{j\neq i}\phi_i(N\backslash\{j\},w|_{N\backslash\{j\}}) = n\phi_i(N,w)$ and by Lemma 3.4, equals $n\left(\frac{1}{2}\phi_i(N,w)+\frac{1}{2}e_i(N,w)\right)$, and then yields $n\gamma_i(N,w)$. Since the payoffs are the same regardless of who is the proposer, the payoff of each player in any equilibrium must coincide with the consensus value. ∎

Mechanism C can be generalized in a natural way by treating the players asymmetrically: bids made by one player are "worth more" than those made by the other. Such a mechanism implements the generalized consensus value of a zero-monotonic partition function form game.

4 Concluding remarks

By using a class of bidding mechanisms that differ in the power awarded to the proposer chosen through a bidding process, this paper provided a strategic approach to several cooperative solution concepts for partition function form games, which highlights the different non-cooperative rationales of the normative standards over externalities. It should be noted that the mechanisms introduced in this paper yield the actual values implemented rather than implementing them in expected terms.

As we see, introducing the option of renegotiation can result in different equilibrium outcomes and therefore implement various values. Throughout the paper we require a renegotiation between a coalition S and a singleton player i to happen only when S has already reached an agreement. If the players in S do not form a coalition, no renegotiation will happen between any of them and i. Thus, in each of the above mechanisms, when the game reaches a level that only involves a sub-coalition of players in N, the other players (i.e., all previously rejected proposers) cannot negotiate among themselves. Without such a restriction, one usually get to a situation where no equilibrium may exist. However, this restriction can be weakened to a certain degree by imposing alternative rules (except for the completely laissez-faire case) on renegotiation. Then it might lead to alternative equilibrium outcomes, hence new values for partition function form games.

The only condition we imposed thus far on the partition function form games is zero-monotonicity, which implies that the grand coalition is the efficient coalition structure. Naturally, one may ask how to deal with arbitrary cooperative environments. Here we like to conclude the paper by suggesting an answer, which is in the same spirit of Pérez-Castrillo and Wettstein (2001). Take the mechanism to implement the Shapley value for example. Given an arbitrary partition function form game, we require a proposer to make a proposal in two aspects: a coalition structure and a vector of offers. Then, if the proposal is accepted, all players will form a coalition structure as specified by the proposer, and the proposer will pay every other player the promised offer. As a reward to the proposer, she will collect all the payoffs generated by each coalition in the coalition structure. Such a modified mechanism generates an efficient coalition structure, and implements the Shapley value of the superadditive cover of the game.

Finally, we note that the results obtained in this paper can shed light on many concrete economic and political situations featured by externalities and cooperative interests and conflicts, ranging from reaching environmental agreement, coordinating market behavior of firms, to the provision of public goods, and resolving the compensation disputes.

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