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Minorities:
Cultural Transmission, Fertility and Integration Policy

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"But the more they were oppressed, the more they multiplies and spread"
(Exodus 1,12)

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Abstract

We live in heterogeneous societies with many cultural and ethnic minorities. The cultural composition of our societies changes over time as a result of fertility choices and cultural assimilation. Studying such population dynamics, we introduce optimal fertility choice into the cultural transmission framework, and examine the effect of integration policies which increase the cost of the direct socialization effort. We show that integration policies, while aimed at reducing the minority's size, may have the opposite effect of increasing minority fertility and its growth rate. We further show that even a one-period increase in the cost of direct socialization may change the long run population structure to one with a large minority.

1. Introduction

The biblical excerpt *"But the more they were oppressed, the more they multiplied and spread"* (Exodus 1, 12) has become central in Jewish culture. It is studied in Jewish schools, referred to in Jewish history classes and is quoted in contemporary political debates. The excerpt describes the Israelites in Ancient Egypt (16^{th} century B.C.) who were oppressed as slave laborers on the one hand, but on the other hand grew in numbers. It does not refer to cultural life – only to the number of people with Israelite identity. The phenomenon described is clearly counterintuitive. Is it possible that oppressing a minority leads to an increase in its size?

The question we study is not just an historical question. We live in heterogeneous societies with many cultural groups. The cultural composition of our societies does change over time, and this may lead to changes in political maps and national identities.

Cultural dynamics are composed of two related processes. The first is a cultural transmission process by which people adopt different cultural identities. Our focus will be on intergenerational cultural transmission, where cultural identity is transmitted from one generation to the next. The second process driving cultural dynamics is the growth of different cultural groups, as determined by their choice of different fertility rates. Combining these processes, parents in our model have two important decisions that may affect the size of their cultural group. The first is their choice of direct socialization effort, i.e. their attempt to influence their children's choice of cultural identity, and the second is their choice of fertility rate.

Often societies wish to affect the assimilation process of minority groups. Some societies would like to discourage it while others would like to speed it up. There are many examples of "Assimilation Policies" or "Integration Policies". While in the ancient world (as well as in some places even today) physical oppression was commonly used in order to affect the social and cultural fabric of society, in present times the emphasis is on legislation promoting different forms of integration policies. Having special schools dedicated to minority groups, educational programs, summer camps, newspapers, theaters and a language are all necessary for a minority group to maintain their cultural identity and transmit it to the next generation. Government integration policies may target these institutions in order to promote social and cultural cohesion. For example, governments sometimes restrict the teaching or usage of minority languages (e.g., the Russian language in Estonia, the Basque language in the Basque province in Spain etc)[1]. In other cases governments may impose restrictions or eliminate subsidies for minority schools or cultural clubs, and may even prohibit certain cultural rituals or dress codes.

The focus of this paper is on the conditions under which the phenomenon described in Exodus 1, 12 arises, and integration policies that make cultural transmission more difficult[2] prove ineffective, resulting in even larger minority groups. [3]

Our basic setup introduces fertility choice into the dynamic cultural transmission framework presented by Bisin and Verdier (2001).[4] In their basic setting individuals have paternalistic altruistic preferences - they care about the cultural identity of their children. Cultural identity is transmitted between generations via a two stage socialization process. The first stage a direct socialization process, where parents' choice of investment in each child determines the probability that the child inherits the parents' cultural identity. Whenever direct socialization fails, children are subject to "oblique socialization", where they adopt the cultural identity of a role model chosen at random from the entire population. The long-run relative size of the cultural groups is the steady state of this two-phase socialization process. Bisin and Verdier (2000, 2001) identify the conditions for having a stable population with a non degenerate cultural structure.

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[1] For example a governmental integration program in Estonia mandated that all Russian-speaking schools teach at least 60% of their curriculum in the Estonian language, starting the 2007/8 school year. See Krimpe (2001).

[2] There are clearly other forms of integration policies that are aimed to affect the relationship between cultural groups, the acceptance of cultural minorities or the attitude of minorities towards the majority group. As it was documented by Joppe (2009) there are also limits to such integration policies.

[3] An interesting example is given by Clark (1981) that documented that the Basque Nationalist Party was growing only slightly in areas where Basque is widely spoken, but is making significant gains in area of low usage.

[4] This population dynamics is based on earlier work in Anthropology by Cavalli-Sforza and Feldman (1973, 1981) and Boyd and Richardson (1985). An alternative setup for population and preferences dynamics would be based on evolutionary sociobiology see Becker (1970), Dawkins (1976) and Frank (1987).

Yet the size of each cultural group depends also on the fertility choices of its members. Different cultural groups may have different preferences regarding the number of children they have. Our focus is on endogenous fertility decision in which fertility choice depends on the expected cultural identity of children as well as on the type of "paternalistic altruism" individuals have.[5]

When individuals have several children "paternalistic altruism" may have different forms. It is not enough to say that individuals prefer that their children will be of their own type. In some cultural groups the emphasis is on having at least one child that remains loyal to the same cultural group – we denote this type as a "survivalist type". In other cultural groups the emphasis is on having all children maintain the group's cultural identity and there may even be a collective penalty on families in which even one of their children "betrays" the group and adopts a different identity – we denote this type as a "zealous type".[6] Most cultural groups have paternalistic preferences that are in between these two extreme types and the paper considers the effect of different types of paternalistic preferences on parents' choice of fertility rate and direct socialization effort, and consequently on the size of the minority group.

When a government introduces an integration policy that makes direct socialization more costly individuals tend to reduce their socialization effort. If this is the only effect then such an integration policy would indeed reduce the size of the minority cultural group. But the paper shows that, depending on the type of paternalistic preferences and the size of the minority group, individuals may react to such an integration policy by increasing their fertility rate. The paper identifies the conditions under which an integration policy, while aimed at reducing the minority's size, may have the opposite effect of increasing minority fertility and its growth rate. The paper also investigates the stable population structure and shows that it may depend on the initial size of the minority and majority groups. Furthermore, the stable population structure may be affected by an integration policy such that even a one-

[5] The endogenous fertility of individuals has been extensively discussed in the literature (for surveys see Easterlin (1978) and Nerlove, Razin and Sadka (1988)).

[6] For example in the Jewish Orthodox community if one of the children becomes secular the marriage prospects for all of his/her siblings is affected.

period increase in the cost of direct socialization may change the long run population structure to one with a larger minority.

2. A Model of Cultural Transmission with Fertility Choice

Our model is based on the cultural transmission model of Bisin and Verdier (2000, 2001) into which we introduce endogenous fertility. Consider a society which consists of two uneven cultural groups, the minority group and the majority group denoted by r and m respectively. The fraction of the minority in the population is denoted by $q_r \in (0, 0.5)$ (and $q_m = 1 - q_r$). We consider an overlapping-generations model in which each individual lives for two periods. In the first period, the childhood period, cultural identity is determined. In the second period, adulthood, individuals bear children and engage in socialization activities. Each adult chooses a fertility rate and his socialization effort. We assume for simplicity that individuals may have either one or two children.

We assume that individuals have paternalistic altruistic preferences such that each individual would like his children to be of his own type (or belong to his own cultural group).[7] When an individual of type $i \in \{r, m\}$ has only one child we let $V_i(j)$ be his utility from having a child of type $j \in \{r, m\}$. This utility includes all the costs of child bearing, as well as the joy of having a child, and can therefore be either positive or negative. We let $\beta = V_r(r) - V_r(m)$ where $\beta > 0$ be the value of having one child who maintains his parent's type for a minority group member.

When people have two children their paternalistic altruistic preferences may take different forms. In some societies the emphasis is on having at least one child who maintains the parent's type, while other in other societies there is a strong emphasis on having all children maintain their parent's type. These societies impose a collective penalty on a family whenever one of its children deviates to the majority type. We let $V_i(j_1 j_2)$ be the value for type i individual of having children of types j_1 and j_2, where $i, j_1, j_2 \in \{r, m\}$. Focusing on the minority's preferences,

[7] Clearly there are counter examples in which a group of new immigrants would like their children to be assimilated into the larger society.

paternalistic preferences imply that $V_r(rr) \geq V_r(rm) \geq V_r(mm)$. We let $B \equiv V_r(rr) - V_r(mm)$ and define $\delta \in (1,2]$ such that $B = \delta\beta$. We further use the normalization $V_r(m) = V_r(mm)$, while assuming $V_r(rr) > V_r(r)$.[8]

We distinguish between two types of paternalistic preferences:

(i) **Survivalist type**: Would like to have **at least** one child of his own type. We capture such preferences by assuming that for this type $V_r(rr) = V_r(rm)$.

(ii) **Zealous type**: Would like to have **all** of his children maintain his type. Having a child that "converts" to the other type imposes a large penalty. For this type we assume that $V_r(rm) = V_r(mm)$.

We assume that the general paternalistic preferences are in between the above distinct types. We let $\mu \in [0,1]$ be the proportion of *B* which is "lost" from having one of the two children deviate to the opposite type. That is, $V_r(rr) - V_r(rm) = \mu B$ (and respectively $V_r(rm) - V_r(mm) = (1-\mu)B$). Therefore a survivalist type is represented by $\mu = 0$ while a zealous type is represented by $\mu = 1$. We refer to μ as the group's level of zealousness.

Cultural Transmission: Cultural transmission may take one of two forms. The first is **direct socialization** which occurs either inside the family or at special schools. This form of socialization is an outcome of parental effort. We denote the degree of direct socialization of individuals of group *i* (where $i \in \{m, r\}$) by τ_i, which represents the probability that a child of a type *i* parent becomes type *i* through the process of direct socialization. We assume that the cost of such direct socialization is $\alpha\tau_i^2 / 2$. We further assume that the direct socialization effort is determined at the household level and that there is no possibility to discriminate among children. Therefore

[8] We need to normalize the utility from having one child and two children as the fertility decision will be based on such a comparison. We therefore assume $V_r(m) = V_r(mm)$.

whenever an individual has two children the socialization effort is identical for both children.[9]

Children whose cultural type has not been determined via the direct socialization process are subject to **oblique socialization** - they randomly choose a role model from the entire population, and adopt the cultural identity of this role model.

Denote by $p_i(j)$ the probability that a child of type i parent becomes type j individual (where $i, j \in \{m, r\}, i \neq j$):

$$(1_a) \quad p_i(i) = \tau_i + (1-\tau_i)q_i$$

$$(1_b) \quad p_i(j) = (1-\tau_i)(1-q_i)$$

When there are two children the same process of cultural transmission applies independently for each child. Therefore:

$$(1_c) \qquad p_i(j_1 j_2) \equiv p_i(j_1) \cdot p_i(j_2), \text{ where } j_1, j_2 \in \{m, r\}.$$

We begin by examining the direct socialization effort for each fertility choice, and then proceed to find the optimal fertility choice.

2.1 The Direct Socialization Choice

The choice of direct socialization effort for a member of the minority group depends on the number of children he has. We derive the optimal direct socialization effort, with one and two children.

One child: The utility of a minority group member from having one child and choosing the direct socialization level τ_r is denoted by $u_r^1(\tau_r \mid q_r)$ and given by:

$$(2) \quad u_r^1(\tau_r \mid q_r) \equiv [\tau_r + (1-\tau_r)q_r]V_r(r) + (1-\tau_r)(1-q_r)V_r(m) - \frac{1}{2}\alpha\tau_r^{\,2}.$$

Maximizing (2) with respect to τ_r yields

$$(3) \quad \tau_r^1(q_r) = \frac{(1-q_r)(V_r(r) - V_r(m))}{\alpha} = \frac{1-q_r}{\alpha}\beta.$$

[9] This assumption can be easily modified.

We assume that $\alpha > \beta$, which guarantees that $0 \le \tau_r^1(q_r) < 1$.

Two children: The expected utility of an individual of a type μ minority, who has two children, is denoted by $u_r^2(\tau_r \mid q_r, \mu)$ and given by[10]:

$$(4) \quad \begin{aligned} u_r^2(\tau_r \mid q_r, \mu) = {} & [\tau_r + (1-\tau_r)q_r]^2 V_r(rr) + (1-\tau_r)^2(1-q_r)^2 V_r(mm) + \\ & 2(1-\tau_r)(1-q_r)[\tau_r + (1-\tau_r)q_r]V_r(rm) - \alpha\tau_r^2. \end{aligned}$$

Maximizing (4) yields the optimal direct socialization level

$$(5) \quad \tau_r^2(q_r, \mu) = \frac{B(1-q_r)\left[q_r\mu + (1-q_r)(1-\mu)\right]}{\alpha - (1-q_r)^2(2\mu - 1)B}.$$

We further assume that $\alpha > B$, which implies that $0 \le \tau_r^2(q_r, \mu) < 1$.

2.1.1 Cultural Substitution.

Following (3), whenever an individual has one child, increases in q_r reduce the incentives to invest in direct socialization. This is because a higher q_r implies a higher probability that a minority child will pick a minority role model at random from the entire population, in the event that direct socialization fails. This is the "cultural substitution effect" defined and discussed by Bisin and Verdier (2001). Note that this cultural substitution effect is instrumental in the survival of small minority groups. Individuals invest more in direct socialization when their group is small, thus contributing to the survival of their group.

Proposition 1: (i) When individuals have only one child the direct socialization effort τ_r is decreasing with q_r (cultural substitution). (ii) When individuals have two children the cultural substitution property does not necessarily hold. When $\mu > 0.5$ there is a critical $q^e(\mu)$ such that whenever $q_r < q^e(\mu)$ the optimal direct socialization effort increases with q_r.
Proof: See Appendix.

[10] Note that μ enters (4) via $V_r(rm)$ (as $V_r(rm) = V_r(mm) + (1-\mu)B$).

Proposition 1(ii) argues that the cultural substitution effect may not hold when there are two children. The intuition is that when there are two children the probability of having one child of type *r* and one child of type *m*, denoted by $p_r(rm) \equiv p_r(r) \cdot p_r(m)$, is non-monotonic in q_r. While $p_r(r)$ increases in q_r and $p_r(m)$ decreases in q_r, $p_r(rm)$ increases for low values of q_r and decreases for high values of q_r. This implies that for low levels of q_r, an increase in q_r results in a relatively higher level of $p_r(rm)$, which increases the incentives for a relatively zealous type to invest in direct socialization, as the biggest gain for such a type is to transform his children from (*rm*) types to (*rr*) types. Thus for a small and zealous minority the cultural substitution effect does not hold. However, when individuals are relative survivalists (μ is small) or when the minority is large and zealous (both μ and q_r are large) the cultural substitution effect will indeed hold.

Throughout the remainder of the paper we will continue to focus on the minority group, omitting the subscript *r* whenever it does not cause ambiguity.

2.1.2 Direct Socialization and the degree of zealousness.

Does zealousness lead to a higher investment in direct socialization? It seems that the larger the penalty for a deviation of just one of two children, the larger the investment in direct socialization, to ensure that all children maintain their cultural identity. But as Lemma 1 points out this is not necessarily the case.

Lemma 1: When an individual has two children the direct socialization effort increases in the level of zealousness only when the minority is sufficiently large. There exists $q^{\tau}(\mu)$ such that the direct socialization effort increases in μ for $q > q^{\tau}(\mu)$ and decreases otherwise.

Proof: See Appendix. ■

The intuition for the above result is similar to the intuition of Proposition 1(ii). A higher level of zealousness has two opposing effects. On the one hand, as the minority becomes more zealous, preserving both children's type becomes more important. When the minority is large enough, oblique socialization provides a "safety net" against deviation to the majority type, and thus the socialization effort will increase in zealousness. On the other hand, as the level of zealousness increases, individuals have more to lose from one child's deviation. For a small minority, chances of successful oblique socialization are low, and thus costly effort will decrease in the level of zealousness.

2.1.3 Direct Socialization and the number of children

Do parents invest more (per child) in direct socialization when they have one child compared to when they have two children? This question relates to the discussion on the quantity/quality tradeoffs in child education (see for example Becker and Lewis (1973) and Becker and Tomes (1976))..

Lemma 2: Zealous individuals will choose a higher direct socialization effort with two children compared to the direct socialization effort with one child. Specifically, there is a $\mu^e(q)$, such that for every $\mu > \mu^e(q)$ (respectively $\mu < \mu^e(q)$), the optimal direct socialization effort with two children is higher (respectively lower) than the optimal effort with one child.
Proof: See Appendix.

When a parent is relatively zealous (high μ) and one child maintains the parent's type, the marginal utility from the type of the second child is high and increases with the parent's zealousness. Thus a zealous minority will choose a higher effort with two children, compared to the effort with one child. The threshold level $\mu^e(q)$ increases in q when the cost of direct socialization is not too high (see the Appendix for specific expressions). This is due to a strong cultural substitution effect with two children, when both q and μ are large. Alternatively, when the cost of effort is large, $\mu^e(q)$ decreases in q, resulting in a larger domain of μ's for which a two-child minority applies a higher effort. This is because when α is large the cultural substitution effect is stronger for a one-child minority[11].

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[11] Lemma 2 and subsequent results rely on the assumption that μ and α are independent. Alternatively, we could assume that α decreases in μ, as a zealous minority would invest more in public goods, such as cultural institutions. Specifically, for proposition 2, assuming that α decreases in μ, would result in a lower threshold $\mu^e(q)$.

2.2 The Fertility Choice

We now turn to consider the individuals' fertility decision. Let $U^1\left(q\right)$ denote the expected utility of a minority group member from having one child and choosing the optimal direct socialization level $\tau^1(q) = (1-q)\beta / \alpha$.

$$\text{(6)} \quad U^1(q) = \frac{\beta^2}{2\alpha}(1-q)^2 + \beta q + V_r(m).$$

Similarly let $U^2(q,\mu)$ be the expected utility of a minority member type μ that has two children and chooses the optimal direct socialization level $\tau^2(q,\mu)$.

$$\text{(7)} \quad U^2(q,\mu) = \frac{B^2(1-q)^2[q\mu + (1-q)(1-\mu)]^2}{\alpha - (1-q)^2(2\mu - 1)B} + qB[q\mu + (2-q)(1-\mu)] + V_r(mm)$$

Individuals' fertility decisions are based on comparing the utilities in (6) and (7). In Figure 1 we describe the utility from having one child and from having two children as a function of q. Maintaining our focus on the minority group, we restrict attention to $q < 0.5$. We graph the utility for three levels of zealousness: the zealot type ($\mu = 1$), the survivalist type ($\mu = 0$) and for an intermediate type ($\mu \in \left(\mu^L, \mu^H\right)$, where μ^L and μ^H are defined in proposition 2).

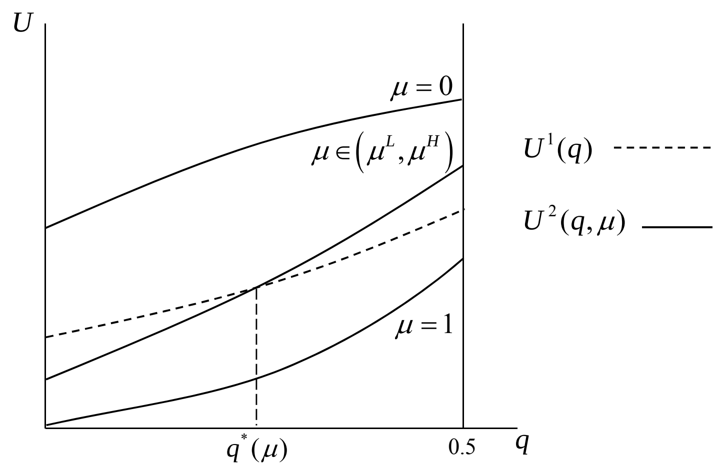


Figure 1: The utility from one vs. two children

When individuals are relatively survivalists (low μ), they suffer little or no utility loss when one of two children deviates to the majority's type. Thus a survivalist type will always prefer having two children regardless of the minority's size. For zealous types (high μ), this argument is reversed as they suffer a large utility loss if one of their two children switches to the majority's type. Therefore zealous types will prefer having one child, to minimize the risk of deviation by one of their children. This also implies that the utility from two children is decreasing in the level of zealousness, μ.

When individuals have moderate paternalistic preferences (mid-range μ), their fertility choice will depend on the minority size which determines the probability of beneficial oblique socialization. This implies that for a relatively large q, moderate types will choose two children, and for a small q, they will choose to have only one child. Furthermore, an increase in the level of zealousness increases the range of minority sizes for which individuals choose one child. This is summarized in Proposition 2.

Proposition 2: (i) Relatively zealous individuals will choose to have one child regardless of q whereas relative survivalists will always choose to have two children. (ii) For moderately zealous types the fertility choice will depend on the minority's size. When q is small, individuals will have one child and when it is large they will prefer to have two children. Formally, there are μ^L, μ^H such that for $\mu \leq \mu^L$ two children are chosen for all q, and for $\mu \geq \mu^H$ one child is chosen for all q. Whenever $\mu^L < \mu < \mu^H$ there exists $q^*(\mu,\alpha) < 0.5$, such that two children are chosen for $q > q^*(\mu,\alpha)$ and one child is chosen for $q \leq q^*(\mu,\alpha)$. Furthermore, $q^*(\mu,\alpha)$ is increasing in μ.
Proof: See appendix. ■

2.3 Integration Policy and the Optimal Fertility Choice

Integration policy is a policy that aims to reduce the size of the minority group. In our model, an obvious candidate for such a policy is any policy that makes direct socialization more difficult or more costly, i.e. any policy which increases α. This can be done by reducing or eliminating government subsidies for special minority clubs, books, newspapers, theater and other cultural activities that are instrumental in maintaining the minority's identity.

Definition 1: We label a public policy α as *a policy that encourages minority fertility* for minority type μ if $\partial q^{*}(\mu,\alpha)/\partial\alpha < 0$, i.e. such a policy increases the range of q for which minority members prefer to have two children.

When the fertility choice is held fixed an increase in the cost of direct socialization will lower the optimal socialization effort, thereby decreasing the size of the minority group in the following period. The question however, is whether raising the cost of direct socialization could alter the minority's fertility rate, and specifically whether an α -increasing policy could result in a higher fertility rate, thus defying its original purpose.

Proposition 3 (Integration policy and fertility rate): Increasing the cost of direct socialization, α, may encourage minority fertility. Formally, there exists $\mu^{\alpha}(q)$, such that for minority type μ; $\mu < \mu^{\alpha}(q)$ and $\mu \in (\mu^{L}, \mu^{H})$, increasing α decreases $q^{*}(\mu,\alpha)$.
Proof: See appendix. ■

The intuition is that for a given α, a relatively survivalist individual (i.e., $\mu < \mu^{\alpha}(q)$) would prefer to have one child and choose a high direct socialization effort in order to have a high probability that his child will retain his minority identity. But when direct socialization becomes more expensive this individual would prefer to have two children and choose a low direct socialization effort, hoping that at least one of the two children would maintain the minority identity.

Figure 2 illustrates the intuition for proving this result. Both the optimal socialization effort and the resulting utility for each fertility choice decline as the cost of direct socialization increases. When two children are chosen, the level of zealousness mitigates the effect of an increase in cost. Thus, for a low level of zealousness both the decline in the optimal effort and the resultant utility when having two children is relatively lower compared to the decline in effort and utility with one child. Consequently, when the minority is not too zealous, an increase in the cost of effort may cause individuals to switch from having one to two children. Note that this switch occurs only when $q \in \left(q^*(\mu,\alpha'), q^*(\mu,\alpha)\right)$, where α' is the new higher cost of direct socialization ($\alpha' > \alpha$).

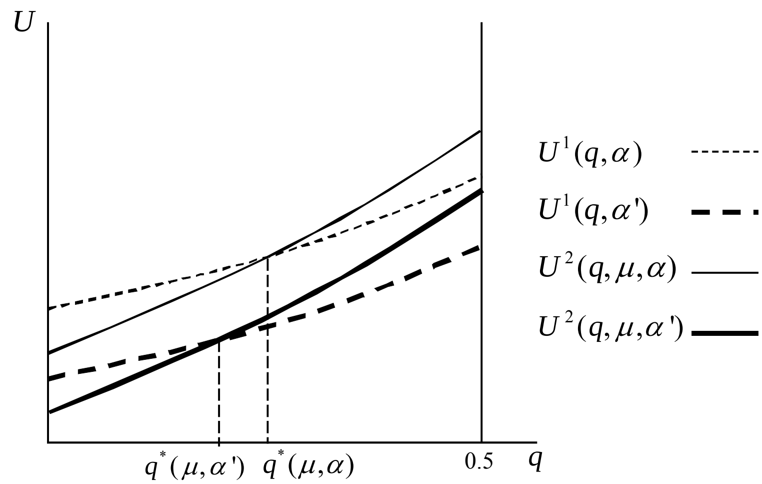


Figure 2: The utility from one vs. two children, before and after an increase in the cost of direct socialization ($\alpha' > \alpha$), for an α-policy that encourages minority fertility.

Numerical Example: Consider the case described in Figure 2. Parameter values are $\alpha = 2,\ \beta = 1$ and δ=1.4. For this case $\mu^\alpha(q) \approx 0.7$, and thus whenever $\mu^L \le \mu < 0.7$ there exists a range of minority sizes for which integration policy will induce an increase in the fertility rate. For example, whenever $\mu = 0.65$ and the minority group is 26.5% of the population, members of the minority group choose to have one child when $\alpha = 2$. Increasing the cost of direct socialization by 10% to $\alpha' = 2.2$ will induce a switch to two children.

3. Minority Growth Rate

The growth rate of each cultural group is determined by its fertility rate and the cultural transmission process, as well as by those of the other group. The resultant population dynamics will evolve in the following way:

$$(8)\quad q_r^{t+1} = \frac{q_r^t N_r^t p_r^t(r) + q_m^t N_m^t p_m^t(r)}{q_r^t N_r^t + q_m^t N_m^t} \ .$$

Where q_i^t is the fraction of group $i \in \{r,m\}$ in the population in period *t*. $p_r^t(r)$ and $p_m^t(r)$ are respectively the probability that a minority child maintains his type and the probability that a majority child would switch to the minority's type (see eqs. (1_a) and (1_b)). The fertility choice of group $i \in \{r,m\}$ in period *t* is denoted by $N_i^t \in \{1,2\}$. This implies that the size of the population in period *t+1* relative to period t, is given by $q_r^t N_r^t + q_m^t N_m^t$.

In this section we examine one-step dynamics, i.e. the ratio q_r^{t+1} / q_r^t. One-step dynamics are of particular interest as a period in our dynamic process is a generation and therefore converging to the long run steady state may take many generations. One-step dynamics are thus the main interest of a policy maker, who wishes to assess the effect of different policies on the minority's expected growth rate from one generation to the next.

Maintaining our focus on the minority's decisions we assume that members of the majority group have only one child and apply no direct socialization effort, relying solely on oblique socialization. We start by stating the conditions under which a low fertility minority would enjoy a faster growth rate than a high fertility minority.

Lemma 3: There exist $q^g(\alpha,\beta,\delta)$ and $\mu^g(q,\alpha,\beta,\delta)$ such that a minority in which individuals have only one child would grow faster than a minority in which individuals have two child whenever:

(i) $q < q^g(\alpha,\beta,\delta)$ and $\mu > \mu^g(q,\alpha,\beta,\delta)$, or –

(ii) $q > q^g(\alpha,\beta,\delta)$ and $\mu < \mu^g(q,\alpha,\beta,\delta)$

Proof: See Appendix. ■

Simply stated, a one-child minority will grow faster than a two-child minority, when its members optimally choose a significantly higher level of direct socialization effort when having one child. In these cases the effect of a high direct socialization effort overrides the effect of higher fertility. This is characteristic of a small and zealous minority who will invest heavily in one child, but also of a large and survivalist minority, which will put in only a little effort whenever choosing to bear two children, relying on oblique socialization. Alternatively, note that a high fertility minority will grow faster than a low fertility minority whenever the effect of the fertility choice prevails. Thus a small and survivalist minority will apply low levels of effort with both one and two children, and will grow at a faster rate when choosing to have two children. Finally, note that for a large and zealous minority, the two effects are aligned. This minority will choose a high fertility rate along with high levels of direct socialization effort, which will result in a higher growth rate when two children are chosen.

3.1 The effect of minority's zealousness level on its growth rate

Combining the result on fertility choice (Proposition 3) with the result on growth rate (Lemma 3), we can now discuss the effect of minority zealousness on its growth rate.

Figures 3a and 3b graph the minority's growth rate as a function of its relative size for two minority types, μ_1 and μ_2; $\mu_1 < \mu_2$. The figures differ in the maximal benefit from two children, δ. When δ is sufficiently large, individuals apply a higher effort, and the growth rate with two children is higher than the growth rate with one child, for all minority sizes. However, when δ is small, individuals apply a smaller effort with two children, and there exists a range of minority sizes for which a two-child minority will exhibit a lower growth rate than a one-child minority.

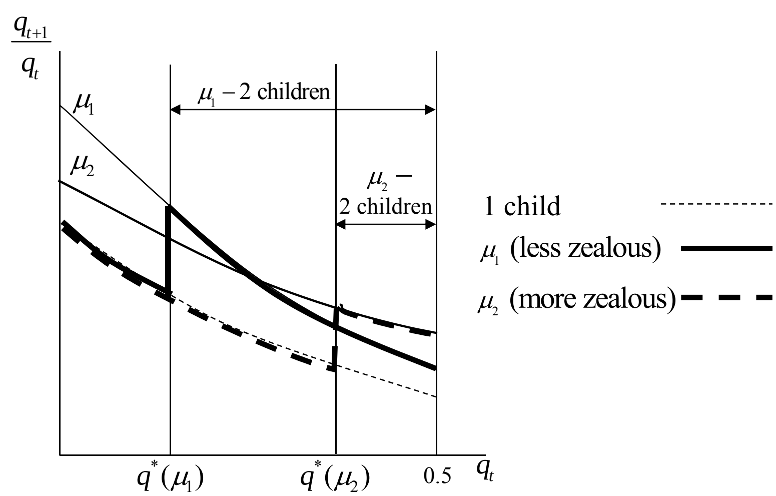


Figure 3a: Minority growth rate as a function of its size and fertility choice, for two minority types $\mu_1 < \mu_2$.

Figure 3a presents the case in which the fertility effect dominates the direct socialization effect and consequently a two-child minority grows faster than a one-child minority (Lemma 3 does not hold). For this case, we compare the growth rate of the two minority types, for three ranges of q:

(i) $q \le q^*(\mu_1)$ - both types choose to have one child and consequently have the same growth rate.

(ii) $q \in \left(q^*(\mu_1), q^*(\mu_2)\right]$ - only the less zealous minority (characterized by μ_1) chooses to have two children while the more zealous minority (characterized by μ_2) chooses to have only one child. In this case the less zealous minority grows faster.

(iii) $q > q^*(\mu_2)$ - Both minority types will choose to have two children, and the more zealous minority grows faster due to a higher level of direct socialization effort.

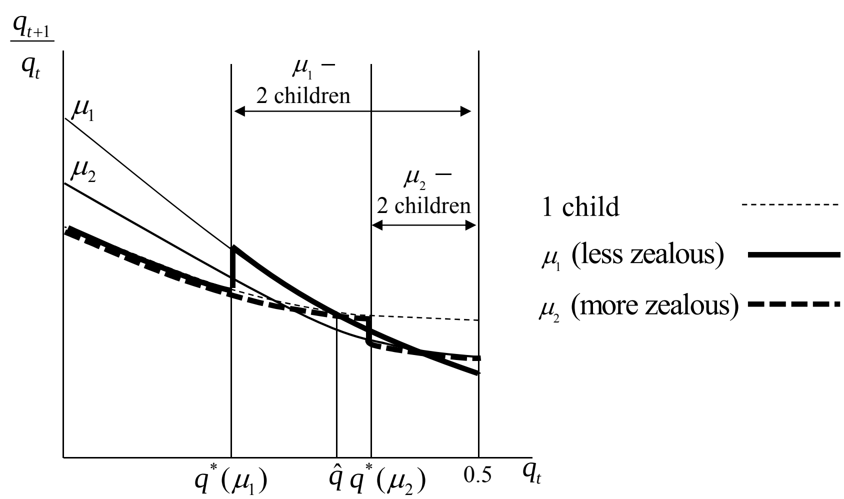


Figure 3b: Minority growth rate as a function of its size and fertility choice, for two minority types $\mu_1 < \mu_2$.

Figure 3b demonstrates the case in which a two-child minority will not necessarily enjoy a faster growth rate than a one-child minority, due to the lower direct socialization effort applied with two children as described in Lemma 3. Again, we compare the growth rate of the two minority types, for three ranges of q:

(i) $q \le q^*(\mu_1)$ - Both types choose to have one child and have the same growth rate.

(ii) $q \in \left(q^*(\mu_1), q^*(\mu_2)\right]$ - Only the less zealous minority (characterized by μ_1) chooses to have two children. There is a $\hat{q}$, such that:

a) For $q \in \left(q^*(\mu_1), \hat{q}\right)$ - the less zealous minority has a higher growth rate.

b) For $q \in \left(\hat{q}, q^*(\mu_2)\right]$ - the more zealous minority enjoys a higher growth rate despite its lower fertility, as the direct socialization effect dominates the fertility effect.

(iii) $q \ge q^*(\mu_2)$ - Both minority types choose to have two children. The minority type which applies a higher effort will grow at a faster pace. Following Lemma 1, the more zealous minority applies a higher effort only when the minority's size is large enough.

Proposition 4 (level of zealousness and minority growth rate):
Consider two types of minorities which differ in their level of zealousness μ_1 and μ_2, such that $\mu^L < \mu_1 < \mu_2 < \mu^H$.
(i) When $q \le q^*(\mu_1)$ both types choose one child and their growth rate is the same.
(ii) When $q^*(\mu_1) < q \le q^*(\mu_2)$, the μ_1 minority will choose two children and the μ_2 minority will choose one child. Minority of type μ_1 will typically grow faster, however when the conditions of Lemma 3 are satisfied, the more zealous minority, μ_2, will exhibit a higher growth rate (see Figure 3b).
(iii) When $q > q^*(\mu_2)$ both types choose two children and the more zealous minority will grow faster when the conditions of Lemma 1 are met.

It follows from proposition 4 that an (exogenous) increase in the minority's level of zealousness will result in a higher growth rate, when the minority continues to choose high fertility (two children) and the direct socialization effort increases as a result of the increased zealousness. An increase in the level of zealousness may also induce a shift to a higher growth rate when the increased zealousness leads to a lower fertility rate accompanied by a significantly higher direct socialization effort. In this case, the growth rate will increase even though the fertility rate declines.

Numerical Example: We provide a simple numerical illustration for the effect of minority zealousness on its growth rate. We consider the case for which $\alpha = 2,\ \beta = 1$ and $\delta = 1.5$, and compare the growth rate of a more zealous minority, $\mu_2 = 0.8$, with the growth rate of a minority of type $\mu_1 = 0.6$ (see Table 1). When the size of the minority consists of 25% of the population, the μ_2 minority has only one child while the μ_1 minority has two children and grows faster. On the other hand when the size of the minority is 45% of the population both groups choose to have two children and the μ_2 minority grows faster since its members invest more in direct socialization.

q	μ	Fertility choice	τ	q_{t+1}/q_t
$q = 25\%$	$\mu_1 = 0.6$	2	0.2765	1.332
	$\mu_2 = 0.8$	1	0.3750	1.281
$q = 45\%$	$\mu_1 = 0.6$	2	0.2117	1.161
	$\mu_2 = 0.8$	2	0.2244	1.170

Table 1: Minority zealousness and growth rate.

3.2 The effects of "integration policy" on minority growth rate.

Minority growth is determined by both fertility choice and the direct socialization effort. When integration policy does not change the fertility choice it will always decrease the minority's growth rate, through the decrease in the direct socialization effort. But as Proposition 3 indicates, integration policy may encourage higher fertility. We consider below two cases that illustrate the effect of integration policy on minority growth rate.

Case 1 (Higher growth rate with higher fertility rate): Assume a minority characterized by $\mu < \mu^{\alpha}(q)$ ($\mu^{\alpha}(q)$ is defined in Proposition 3). Figure 4a graphs the minority's growth rate as a function of its relative size, and the optimal fertility choice, for two levels of the direct socialization cost. Let α represent the direct socialization cost prior to the onset of the integration policy, while α', $\alpha' > \alpha$, represents the increased cost of socialization resulting from the integration policy.

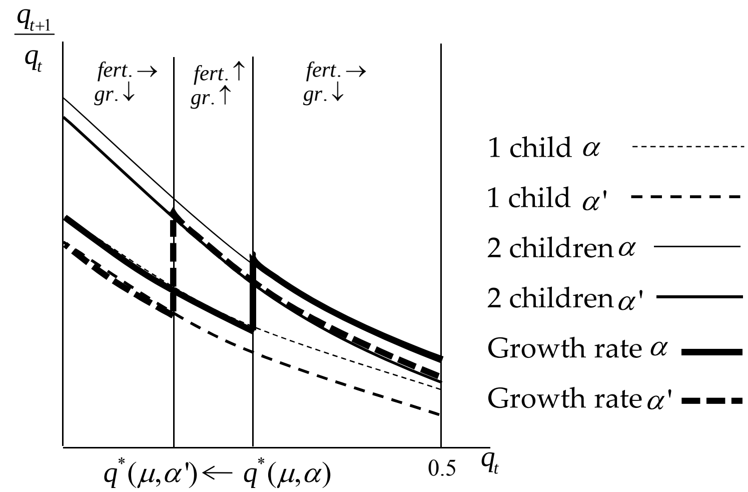


Figure 4a: Minority growth rate as a function of its size and fertility choice, pre and post integration policy- the case of increased growth rate due to higher fertility.

When $\mu < \mu^{\alpha}(q)$ Proposition 3 implies that $q^*(\mu,\alpha') < q^*(\mu,\alpha)$. Thus we analyze three regions of q, as illustrated in the above Figure:

(i) When $q \le q^*(\mu,\alpha')$ integration policy does not affect fertility and members of the minority group choose to have one child. The higher cost of direct socialization makes these individuals reduce their direct socialization effort, resulting in a lower growth rate.

(ii) When $q \in q^*((\mu,\alpha'), q^*(\mu,\alpha)]$ the higher cost of direct socialization induces minority group members to increase their fertility and have two children (see Proposition 3). Thus the higher cost of direct socialization results in a higher minority growth rate.

(iii) When $q > q^*(\mu,\alpha)$ the integration policy does not affect fertility decisions and individuals choose to have two children for both α and α'.

A higher cost of socialization α' leads to lower socialization effort and to a lower growth rate.

Case 2 (Higher growth rate with lower fertility): Assume a minority characterized by $\mu > \mu^{\alpha}(q)$. Figure 4b graphs the minority's growth rate as a function of its relative size, and the optimal fertility choice, for two levels of the direct socialization cost. When $\mu > \mu^{\alpha}(q)$ Proposition 3 implies that $q^*(\mu,\alpha') > q^*(\mu,\alpha)$.

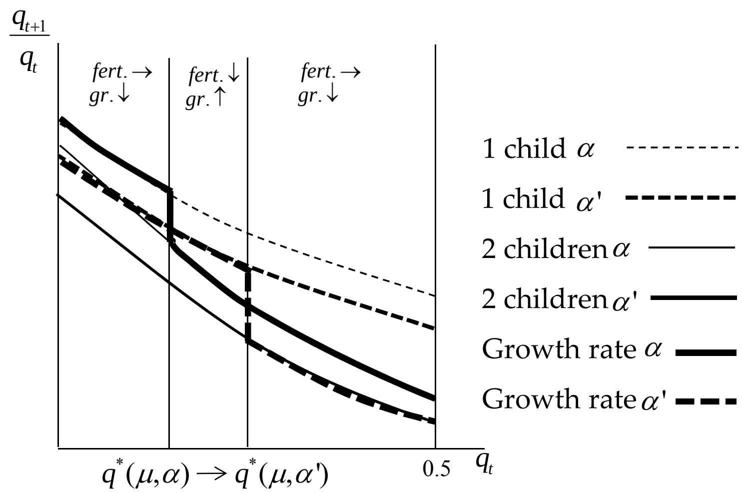


Figure 4b: Minority growth rate as a function of its size and fertility choice, for two levels of socialization cost $\alpha < \alpha'$ - increased growth rate with lower fertility.

Whenever $q > q^*(\mu,\alpha')$ or $q \leq q^*(\mu,\alpha)$ the integration policy does not affect the fertility choice and therefore as in case 1 it results in a lower growth rate due to a lower direct socialization effort. Yet for $q \in \left(q^*(\mu,\alpha), q^*(\mu,\alpha')\right]$ integration policy induces a reduction in the minority's fertility rate from two children to one child (see Proposition 3). Whenever the conditions of Lemma 3 hold, this lower fertility is characterized by a high investment in every only child, leading to a higher growth rate as depicted in Figure 4b.

Proposition 5 (Integration policy and minority growth rate): Increasing the cost of direct socialization, α , will increase the minority's growth rate in the following cases:

(i) When it induces a switch from one to two children and the fertility effect dominates the lower direct socialization effect, such that a two-child minority grows faster (see Figure 4a).

(ii) When it induces a switch from two children to one but with a higher direct socialization effort, and the direct socialization effect dominates the fertility effect, such that a one-child minority grows faster (see Figure 4b).

Numerical Example: To illustrate the effect of integration policy on the minority growth rate we provide a numerical example in which in the base case is $\alpha = 2$ and we examine the effect of a 10% increase in the cost of socialization to $\alpha' = 2.2$ (see Table 2). When the minority is 10% of the population the integration policy does not affect fertility and minority members choose to have one child. The higher costs of socialization induce a reduction in the direct socialization effort which results in a lower growth rate. A similar effect occurs when the minority is 30% of the population, and members of the minority choose to have two children. The interesting effect occurs for a mid-range minority. For example when the minority is 17.5% of the population the integration policy results in a higher fertility rate, lower direct socialization effort and a higher growth rate.

q	α, α'	Fertility choice	τ	q_{t+1} / q_t
$q = 10\%$	$\alpha = 2$	1	0.4500	1.405
	↓ $\alpha' = 2.2$	1	0.4091	1.368
$q = 17.5\%$	$\alpha = 2$	1	0.4125	1.340
	↓ $\alpha' = 2.2$	2	0.2500	1.351
$q = 30\%$	$\alpha = 2$	2	0.2420	1.261
	↓ $\alpha' = 2.2$	2	0.2185	1.235

Table 2: Integration policy and growth rate ($\beta = 1$, $\delta = 1.4$ and $\mu = 0.6$)

4. Stable population structures

We turn to discuss the effect of integration policy on the long-run structure of the population. For the analysis of the long run stable population structure we assume that both the minority and majority group members choose their direct socialization effort and fertility level optimally.[12] We assume however that the majority group is characterized by $\mu_m = 0.5$, i.e., they are neither zealous nor survivalist in their paternalistic preferences. Using (8), a stable minority size, denoted by q^s, satisfies the following steady state condition:

$$(9)\ q^s = \frac{q^s N_r p_r(r) + \left(1-q^s\right) N_m p_m(r)}{q^s N_r + \left(1-q^s\right) N_m}$$

Members of the majority and minority groups may differ in their child-bearing costs and benefits. Throughout this section group subscripts will be added to the cost and benefit parameters, now denoted by α_i and δ_i, for $i \in \{r, m\}$.[13] The only symmetry between the groups is assumed with respect to β, the utility loss when having one child that switches types, i.e., $\beta = V_r(r) - V_r(m) = V_m(m) - V_m(r)$.

Following Proposition 2 we let $q_r^*\left(\mu_r, \alpha_r, \delta_r\right)$ be the threshold of the minority size for which the members of the minority group switch from choosing one to two children (when it is characterized by $\mu_r^L < \mu_r < \mu_r^H$). We now turn to consider the fertility decision of the majority group. Since $q_m > 0.5$ and $\mu_m = 0.5$ the majority's fertility choice depends on δ_m.

Lemma 4: There exists δ_m^* such that for $\delta_m \geq \delta_m^*$ members of the majority group chooses $N_m = 2$ regardless of its size. When $\delta_m < \delta_m^*$, there exists a threshold $q_m^*\left(\mu_m, \alpha_m, \delta_m\right)$ such that when $\delta_m < \delta_m^*$ and $q_m \geq q_m^*\left(\mu_m, \alpha_m, \delta_m\right)$, members of the majority group choose $N_m = 2$, and only when $\delta_m < \delta_m^*$ and $q_m < q_m^*\left(\mu_m, \alpha_m, \delta_m\right)$ they choose $N_m = 1$.
Proof: See Appendix. ■

Note that while, in principal, the majority group may choose to have one child, this would occur only for very low values of δ_m. For most parameter values that we examined δ_m^* is close to 1. A choice of high fertility seems intuitive for the majority

[12] In our analysis of one-step dynamics we assumed that the majority's fertility choice is fixed at one child, and that majority group members do not apply any direct socialization effort.
[13] Recall that $B_r = \delta_r \beta$.

group, as its type of $\mu_m = 0.5$ implies a relatively low penalty for a deviation of one child, which, combined with the group's high probability of successful oblique socialization, results in an overall low child-bearing risk. We will therefore restrict our analysis hereinafter to the case of $N_m = 2$.

We now proceed to study stable population structures. Denote a fertility choice combination by (N_r, N_m), where $N_m \equiv 2$ and $N_r \in \{1,2\}$. We begin by identifying the stable population structure, in terms of the minority's fraction in the population, when (N_r, N_m) is fixed. A stable population structure given (N_r, N_m), is denoted by $q^s_{N_r,N_m}$. We then endogenize the fertility choice to analyze the steady state population structure.

Lemma 5: For each fertility choice pair $(N_r, 2)$ there exists a unique stable steady state population structure, with $q^s_{N_r,2} \in (0,1)$:

(i) For $(1,2)$: $q^s_{1,2} = \alpha_m / (\alpha_m + \delta_m \alpha_r)$, thus $q^s_{1,2} < 0.5$ iff $\alpha_m / \delta_m < \alpha_r$.

(ii) For $(2,2)$: $q^s_{2,2} \in (0,1)$. $q^s_{2,2} < 0.5$ iff $\mu < \frac{1}{2} + 2\left(\frac{\alpha_r}{\delta_r \beta} - \frac{\alpha_m}{\delta_m \beta}\right)$.

Proof: See appendix. ■

As intuition suggests, when minority group members have one child, the group's long-run fraction in the population equals the ratio of the majority's child bearing cost-to-benefit ratio and the sum of both groups' cost-to-benefit ratios, as this ratio determines the group's relative direct socialization effort. Thus, when the minority has a higher child cost-to-benefit ratio, it will converge to a smaller size. When the minority has two children, the resulting population structure will depend on the group's level of zealousness, as well as on the difference in cost-to-benefit ratios between the groups. A relatively survivalist minority will maintain its minority status, and the range of μ for which this occurs increases in the child cost-to-benefit difference in favor of the majority.

Combining our result on the minority's fertility choice (see Proposition 2) with Lemma 5, and assuming that $\alpha_m / \delta_m < \alpha_r$ and $\mu < 0.5 + 2(\alpha_r / \delta_r \beta - \alpha_m / \delta_m \beta)$, such that the minority maintains its minority status in the steady state, allows us to identify the stable steady state population.

Proposition 6: The population will converge to one of the stable structures $q^s_{1,2}$ or $q^s_{2,2}$ (defined in Lemma 5), depending on the initial population structure q^0:

(i) When $\mu_r \le \mu^L$ the population will converge to $q^s_{2,2}$.

(ii) When $\mu_r \ge \mu^H$ the population will converge to $q^s_{1,2}$.

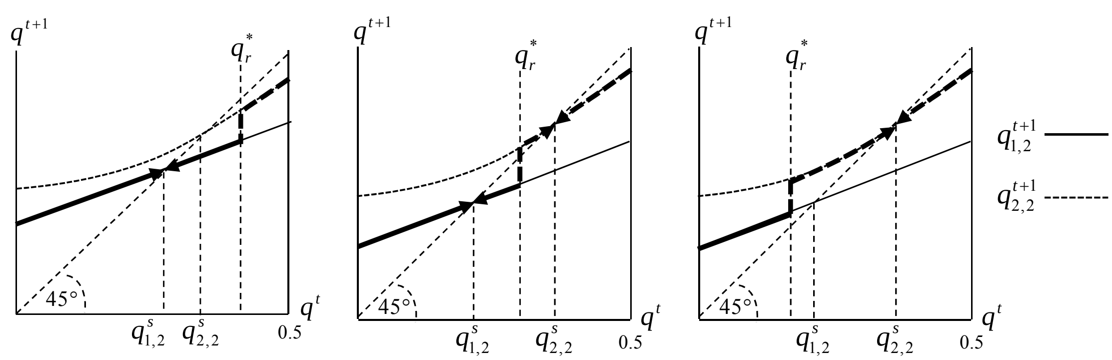
(iii) When $\mu^L < \mu_r < \mu^H$:

a. If $q^s_{1,2} < q^s_{2,2} \le q^*_r(\mu_r, \alpha_r, \delta_r)$ the population will converge to $q^s_{1,2}$.

b. If $q^s_{1,2} \le q^*_r(\mu_r, \alpha_r, \delta_r) < q^s_{2,2}$ the steady state will depend on the initial population structure q^0. If $q^0 \le q^*_r(\mu_r, \alpha_r, \delta_r)$ the population will converge to $q^s_{1,2}$ and if $q^0 > q^*_r(\mu_r, \alpha_r, \delta_r)$ the population will converge to $q^s_{2,2}$.

c. If $q^*_r(\mu_r, \alpha_r, \delta_r) < q^s_{1,2} < q^s_{2,2}$ the population will converge to $q^s_{2,2}$.

In Figure 6 we illustrate convergence to the possible steady states, for different initial minority size. Figures 6a and 6c correspond to cases (iii)a and (iii)c in proposition 6 (respectively), and Figure 6b corresponds to case (iii)b where the steady state depends on the initial population structure. For this latter case, when the initial minority size is $q^0 \le q^*_r(\mu_r, \alpha_r, \delta_r)$ the population will converge to the $q^*_{1,2}$ steady state and when the initial minority size is $q^0 > q^*_r(\mu_r, \alpha_r, \delta_r)$ the population will converge to the steady state of $q^*_{2,2}$, which is characterized by a larger minority.



(a) High μ_r (b) Intermediate μ_r (c) Low μ_r

Figure 6: Convergence to the steady state, from different initial minority size.

We can now consider the effect of integration policy, i.e., imposing higher α_r, on the stable steady state population structure. Increasing α_r changes all the threshold levels in Proposition 6. We therefore do not describe all the possible cases, but provide one example where the onset of an integration policy results in a new steady state, with a larger long-run minority.

Numerical example: Consider the case $\alpha_r = 4,\ \alpha_m = 2,\ \beta = 1,\ \delta_r = \delta_m = 1.5$, illustrated in Figure 6b. For these parameter values, the threshold minority size above which the minority chooses high fertility is 27.5% of the population. Therefore a minority which is just below the threshold and consists of 27.4% of the population will converge to the steady state of $q_{1,2}^s$, with a long run population size of 25%. Following the onset of an integration policy which increases the minority's cost of direct socialization by 10%, the threshold $q_r^*(\mu_r, \alpha_r, \delta_r)$ decreases to 27.3%. For the above minority, this implies a change from low to high fertility, and will thus result in a shift to convergence to the higher steady state of $q_{2,2}^s$ with a minority size of 29% of the population. We can therefore conclude:

Proposition 7: Integration Policy may result in a larger minority in the long run. Specifically, whenever $q_{1,2}^s \le q_r^*(\mu_r, \alpha_r, \delta_r) < q_{2,2}^s$, and $q^0 \le q_r^*(\mu_r, \alpha_r, \delta_r)$ the stable steady state is $q_{1,2}^s$. In this case, introducing an integration policy could lower $q_r^*(\mu_r, \alpha_r, \delta_r)$, and may thus increase the minority's fertility rate, thereby leading to convergence to the new steady state $q_{2,2}^s$, with a larger minority.

Concluding Remarks

The focus of our paper has been on the effect of "Integration policy" on the fertility, assimilation process and growth of minority groups. Integration policy is modeled as an exogenous increase in the cost of the direct socialization effort, which could be the outcome of a decrease in government subsidy to the minority's schools, newspapers, theater or other cultural establishments. A naive policy maker would expect such policies to decrease the socialization effort chosen by the minority, and a subsequent decrease in the minority's size. However, taking into account both fertility and effort choices, we show that integration policy could have the opposite effect. Our results therefore suggest that integration policy should be very cautiously employed.

The cultural transmission model assumes only two forms of socialization; direct socialization and oblique socialization, with individuals choosing only the direct socialization effort. The standard cultural transmission framework may be generalized by allowing individuals to affect the oblique socialization level as well. This may be done by introducing different levels of voluntary segregation that individuals may choose. As a result, when the direct socialization process fails, the probability that a child adopts the majority's identity would also be a decreasing function of the segregation level imposed by his parents. Such a framework may give rise to yet another detrimental effect of integration policies, as these policies could result in a more segregated minority.

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Appendix:

Proof of Proposition 1:
Note that subscripts are omitted from this and following proofs whenever it causes no ambiguity.

The optimal effort with one child is given by $\tau^1(q) = \frac{\beta}{\alpha}(1-q)$, and the derivative w.r.t. q is $\frac{d\tau^1(q)}{dq} = -\frac{\beta}{\alpha} < 0$.

The optimal effort with two children is given by

$\tau^2(q,\mu) = \frac{B(1-q)\left[q\mu + (1-q)(1-\mu)\right]}{\alpha - (1-q)^2(2\mu-1)B}$, and the derivative w.r.t. q is given by:

$$\frac{\partial \tau^2(q,\mu)}{\partial q} = B\frac{\alpha\left[-q\mu + (1-q)(3\mu-2)\right] - B(1-q)^2(2\mu-1)\mu}{\left[\alpha - B(1-q)^2(2\mu-1)\right]^2}$$

This derivative is negative if and only if the following inequality holds:

$$\alpha\left[-q\mu + (1-q)(3\mu-2)\right] - B(1-q)^2(2\mu-1)\mu < 0$$

When $\mu \le 0.5$, the above inequality holds for all values of q.
When $\mu > 0.5$, the above inequality holds for

$q > q^e(\mu),$ where $q^e(\mu) = 1 - \frac{\alpha}{\mu B} + \frac{1}{\mu B}\sqrt{\alpha^2 - \alpha\mu B\frac{\mu}{2\mu-1}} < 1.$

Summarizing,

$$\text{For } \mu > 0.5: \ \frac{\partial \tau^2(q,\mu)}{\partial q} \le 0 \text{ iff } q \ge q^e(\mu)$$

$$\frac{\partial \tau^2(q,\mu)}{\partial q} > 0 \text{ iff } q < q^e(\mu)$$

$$\text{For } \mu \le 0.5: \ \frac{\partial \tau^2(q,\mu)}{\partial q} < 0 \ \text{ for all } q \in [0,1].$$

Proof of Lemma 1:
The derivative of $\tau^2(q,\mu)$ w.r.t. μ is given by:

$$\frac{\partial \tau^2(q,\mu)}{\partial \mu} = B(1-q)\frac{-\alpha(1-2q) + B(1-q)^2}{\left[\alpha - (1-q)^2(2\mu-1)B\right]^2}.$$

This derivative is positive if and only if $-\alpha(1-2q) + B(1-q)^2 > 0$.

Solving yields one positive solution, which we denote by $q^\tau(\mu)$:

$$q^\tau(\mu) = 1 + \frac{1}{B}\left(\sqrt{\alpha^2 - \alpha B} - \alpha\right).$$

Thus $\frac{\partial \tau^2(q,\mu)}{\partial \mu} > 0$ iff $q > q^{\tau}(\mu)$ and it can be shown that $q^{\tau}(\mu) \in (0, 0.5)$. ■

Proof of Lemma 2:
We compare the optimal direct socialization effort with one child to the optimal effort with two children. The optimal effort with one child is larger whenever:

$$\frac{\beta}{\alpha} > \frac{\delta\beta\left[q\mu + (1-q)(1-\mu)\right]}{\alpha - (1-q)^2(2\mu-1)\delta\beta}$$

Solving yields the threshold μ^e, given by:

$$\mu^e \equiv \frac{\alpha + \delta\beta(1-q)^2 - \alpha\delta(1-q)}{\alpha\delta q + 2\delta\beta(1-q)^2 - \alpha\delta(1-q)}$$

Such that for $\mu > \mu^e$ the optimal effort with two children is higher, and for $\mu < \mu^e$ it is lower than the effort with one child.
Note that

$$\frac{\partial \mu^e}{\partial q} = \frac{\alpha\delta(2-\delta)(2\beta(1-q)-\alpha)}{\left[\alpha\delta q + 2\delta\beta(1-q)^2 - \alpha\delta(1-q)\right]^2}$$

Therefore, μ^e (weakly) increases in q whenever $\alpha \le 2\beta(1-q)$, and decreases in q otherwise. ■

Proof of Proposition 2:
We write the utility for minority group members for extreme values of q, for each fertility choice:

$$U^1(0) = \frac{\beta^2}{2\alpha} + V_r(m)$$

$$U^1(0.5) = \frac{\beta^2}{8\alpha} + \frac{\beta}{2} + V_r(m)$$

$$U^2(0,\mu) = \frac{B^2(1-\mu)^2}{\alpha - (2\mu-1)B} + V_r(mm)$$

$$U^2(0.5,\mu) = \frac{B^2}{16\alpha - 4(2\mu-1)B} + 0.5B(1.5-\mu) + V_r(mm)$$

Since $U^1(q)$ and $U^2(q,\mu)$ are monotonically increasing in q, we show that there is a single crossing of $U^1(q)$ and $U^2(q,\mu)$ for $q \in (0, 0.5)$, by showing that $U^1(0) > U^2(0,\mu)$ and $U^1(0.5) < U^2(0.5,\mu)$.
The first condition, $U^1(0) > U^2(0,\mu)$, yields the following inequality:

$$\frac{\beta^2}{2\alpha}-\frac{B^2(1-\mu)^2}{\alpha-(2\mu-1)B}>0\,.$$

This inequality is quadratic in μ and yields μ_1,μ_2 such that for $\mu_1\le\mu\le\mu_2$ the inequality holds. Since $\mu_2>1$ and $0<\mu_1<1$, we denote $\mu^L\equiv\mu_1$:

$$\mu^L=\frac{2\alpha B-\beta^2-\beta\sqrt{\beta^2+2\alpha^2-2\alpha B}}{2\alpha B}\,.$$

And $U^1(0)>U^2(0,\mu)$ for all $\mu>\mu^L$.

Substituting $B=\delta\beta,\ \delta\in(1,2]$, we can show that for $\delta>\sqrt{2},\ \mu^L>0.5$, and for $\delta<\sqrt{2},\ \mu^L<0.5$.

The second condition, $U^1(0.5)<U^2(0.5,\mu)$, yields the following inequality:

$$\frac{B^2}{16\alpha-4B(2\mu-1)}+0.5B(1.5-\mu)>\frac{\beta^2}{8\alpha}+0.5\beta\quad.$$

This inequality is quadratic in μ as well, and yields μ_1,μ_2 such that for $\mu<\mu_1$ and $\mu>\mu_2$ the inequality holds. Since $\mu_2>1$ and $0<\mu_1<1$, we denote $\mu^H\equiv\mu_1$:

$$\mu^H=\frac{8\alpha(\alpha+B)-2\alpha\beta-\beta^2-\sqrt{\left[8\alpha(\alpha-B)+2\alpha\beta+\beta^2\right]\left[8\alpha^2+2\alpha\beta+\beta^2\right]}}{8\alpha B}\,.$$

And $U^1(0.5)<U^2(0.5,\mu)$ for all $\mu<\mu^H$.

We have therefore identified the lower and upper bounds for μ:

For $\mu^L<\mu<\mu^H$ there is single crossing of $U^1(q)$ and $U^2(q,\mu)$ for $q\in(0,0.5)$.

Further note that $\mu^L<\mu^H$ is non-empty – a sufficient condition for $\mu^L<\mu^H$ is $B<\frac{(2\alpha+\beta)^2+2\beta^2}{8\alpha}$ or $\delta<\frac{1}{2}+\frac{\alpha}{2\beta}+\frac{3\beta}{8\alpha}$.

Denote the single crossing of $U^1(q)$ and $U^2(q,\mu)$ by $q^*(\mu,\alpha)$. It remains to show that $q^*(\mu,\alpha)$ is increasing in μ. Clearly, $q^*(\mu,\alpha)$ increases in μ if and only if $U^2(q,\mu)$ decreases in μ. We write the derivative of $U^2(q,\mu)$ w.r.t μ:

$$\frac{\partial U^2(q,\mu)}{\partial\mu}=\frac{-2B(1-q)\left[q\alpha+B(1-q)^2(1-\mu)\right]\left[\alpha-B\mu(1-q)\right]}{\left[\alpha-B(1-q)^2(2\mu-1)\right]^2}<0$$

which completes our proof. ■

Proof of Proposition 3:

Increasing α decreases both $U^1(q)$ and $U^2(q,\mu)$. When the effect of α on $U^1(q)$ is stronger $q^*(\mu,\alpha)$ decreases, and there exist minority sizes for which this implies a change in the fertility choice from one to two children. We proceed by identifying the condition for $\frac{\partial U^2(q,\mu)}{\partial \alpha} - \frac{\partial U^1(q)}{\partial \alpha} > 0$:

$$\frac{\partial U(q,\mu,2)}{\partial \alpha} - \frac{\partial U(q,1)}{\partial \alpha} = (1-q)^2 \left[\frac{\beta^2}{2\alpha^2} - \frac{B^2[q\mu+(1-q)(1-\mu)]^2}{[\alpha - B(1-q)^2(2\mu-1)]^2} \right] =$$

$$= (1-q)^2 \underbrace{\left[\frac{\beta}{\sqrt{2}\alpha} + \frac{B[q\mu+(1-q)(1-\mu)]}{\alpha - B(1-q)^2(2\mu-1)} \right]}_{>0} \underbrace{\left[\frac{\beta}{\sqrt{2}\alpha} - \frac{B[q\mu+(1-q)(1-\mu)]}{\alpha - B(1-q)^2(2\mu-1)} \right]}_{\equiv A}$$

We denote $A \equiv \frac{\beta}{\sqrt{2}\alpha} - \frac{B[q\mu+(1-q)(1-\mu)]}{\alpha - B(1-q)^2(2\mu-1)}$.

$\frac{\partial U^2(q,\mu)}{\partial \alpha} - \frac{\partial U^1(q)}{\partial \alpha} > 0$ if and only if $A > 0$. Further arranging yields that $A > 0$ if and only if $\mu < \mu^{\alpha}(q)$, where:

$$\mu^{\alpha}(q) = \frac{\alpha + \delta\beta(1-q)^2 - \sqrt{2}\alpha\delta(1-q)}{2\delta\beta(1-q)^2 - \sqrt{2}\alpha\delta(1-2q)}$$

This completes our proof. ■

Proof of Lemma 3:

We solve $q^{t+1}\big|_{N_r=1} > q^{t+1}\big|_{N_r=2}$:

$$\frac{(1-q)q + q\left[\tau^1 + (1-\tau^1)q\right]}{(1-q)+q} > \frac{(1-q)q + 2q\left[\tau^2 + (1-\tau^2)q\right]}{(1-q)+2q}$$

This reduces to $\tau^1 > \frac{2}{1+q}\tau^2$:

$$\frac{\beta}{\alpha}(1-q) > \frac{2}{1+q} \frac{\delta\beta(1-q)[q\mu+(1-q)(1-\mu)]}{\alpha - \delta\beta(1-q)^2(2\mu-1)}$$

$$\alpha(1+q) - 2\alpha\delta(1-q) + \delta\beta(1-q)^2(1+q) > \mu \underbrace{\left[2\alpha\delta(2q-1) + 2\delta\beta(1-q)^2(1-q)\right]}_{(*)}$$

The expression marked (*) is increasing in q, and there exists $q^g(\alpha,\beta,\delta)$ such that (*) is negative for all $q < q^g(\alpha,\beta,\delta)$ and positive for all $q > q^g(\alpha,\beta,\delta)$.
The threshold value for μ is derived by dividing the LHS of the inequality by (*).
Denote this threshold by μ^g:

$$\mu^g \equiv \frac{1}{2} + \frac{\alpha(1-\delta+q)}{2\alpha\delta(2q-1) + 2\delta\beta(1-q)^2(1+q)} \quad .$$

We conclude that $q^{t+1}\big|_{N_r=1} > q^{t+1}\big|_{N_r=2}$ whenever:

(i) $q < q^g(\alpha,\beta,\delta)$ and $\mu > \mu^g(q,\alpha,\beta,\delta)$, or –

(ii) $q > q^g(\alpha,\beta)$ and $\mu < \mu^g(q,\alpha,\beta,\delta)$. ■

Proof of Lemma 4:
We begin by writing a majority group member's utility from having one child and from having two children (under the assumption that $\mu_m = 0.5$):

$$U_m^1(q) = \frac{\beta^2}{2\alpha_m}\left(1-q_m\right)^2 + \beta q_m + V_m(r)$$

$$U_m^2(q) = \frac{\delta_m{}^2\beta^2}{4\alpha_m}\left(1-q_m\right)^2 + \delta_m\beta q_m + V_m(rr)$$

Two children are chosen whenever:

$$\frac{\delta_m{}^2\beta^2\left(1-q_m\right)^2}{4\alpha_m} + q_m\delta_m\beta + V_m(rr) > \frac{\beta^2}{2\alpha_m}\left(1-q_m\right)^2 + \beta q_m + V_m(r)$$

Arranging yields:

$$\left(\frac{\delta_m{}^2}{2} - 1\right)\frac{\beta^2\left(1-q_m\right)^2}{2\alpha_m} + q_m\left(\delta_m - 1\right)\beta > 0$$

Note that when $\delta_m \geq \sqrt{2}$ the above inequality holds for all q_m, and more generally, there exists a threshold δ_m, denoted $\delta_m^* \in \left(1,\sqrt{2}\right)$ such that when $\delta_m \geq \delta_m^*$ the inequality holds for all q_m and the majority will always choose $N_m = 2$.
Solving the inequality for q_m, we find that one root is positive and smaller than 1 and the other is larger than 1. Denote these roots by $q_1 < q_2$. When $q < q_1$ or $q > q_2$ the inequality holds. We denote the smaller root by $q_m^*\left(\alpha_m, \delta_m\right)$ and conclude that for $\delta_m < \delta_m^*$ and $q_m < q_m^*\left(\alpha_m, \delta_m\right)$ the majority will choose $N_m = 1$, and otherwise $N_m = 2$.
■

Proof of Lemma 5:
The condition for a stable population structure, given a fertility choice pair $\left(N_r, N_m\right)$ is $N_r\tau_r^{N_r} = N_m\tau_m^{N_m}$. We solve this condition for fertility choice pairs $\left(1,2\right)$ and $\left(2,2\right)$:

(i) For $\left(1,2\right)$: $\tau_r^1 = 2\tau_m^2$ implies –

$$\frac{1-q}{\alpha_r}\beta = 2\frac{\delta_m\beta q}{2\alpha_m}$$

Arranging yields $q_{1,2}^s = \dfrac{\alpha_m}{\alpha_m + \delta_m\alpha_r}$. Note that $q_{1,2}^s < \dfrac{1}{2}$ iff $\alpha_m/\delta_m < \alpha_r$.

Furthermore, this steady state is stable, since $q < q_{1,2}^s$ implies $\tau_r^1 > \tau_m^1$, and thus minorities that are smaller than $q_{1,2}^s$ will grow towards this size, while larger minorities will decrease towards $q_{1,2}^s$.

(ii) For $(2,2)$: $2\tau_r^2 = 2\tau_m^2$ implies –

$$\frac{\delta_r \beta (1-q)\left[q\mu + (1-q)(1-\mu)\right]}{\alpha_r - \delta_r \beta (1-q)^2 (2\mu - 1)} = \frac{\delta_m \beta q}{2\alpha_m}$$

Solving for q is equivalent to finding the zeros of the following function:

$$f_{2,2}(q,\mu) = q^3 \delta_r \delta_m \beta (2\mu-1) + q^2 2\delta_r (\alpha_m + \delta_m \beta)(1-2\mu) + q\left[6\alpha_m \delta_r \mu - 4\alpha_m \delta_r - \alpha_r \delta_m + \delta_r \delta_m \beta (2\mu-1)\right] + \\ + 2\alpha_m \delta_r (1-\mu)$$

We find values of $f_{2,2}(q,\mu)$ for $q \in \{0, 0.5, 1\}$:

$f_{2,2}(0,\mu) = 2\alpha_m \delta_r (1-\mu) \geq 0$

$f_{2,2}(1,\mu) = -\alpha_r \delta_m < 0$

$f_{2,2}(0.5,\mu) = \frac{1}{8}\delta_r \delta_m \beta (2\mu - 1) + 0.5\alpha_m \delta_r - 0.5\alpha_r \delta_m < 0$ iff $\mu < \frac{1}{2} + 2\left(\frac{\alpha_r}{\delta_r \beta} - \frac{\alpha_m}{\delta_m \beta}\right)$

By the intermediate value theorem, we can state that there exists a zero in $(0,1)$. Denote this zero by $q_{2,2}^s$. Furthermore, $q_{2,2}^s \in (0, 0.5)$ when $f_{2,2}(0.5,\mu) < 0$, i.e. if and only if $\mu < \frac{1}{2} + 2\left(\frac{\alpha_r}{\delta_r \beta} - \frac{\alpha_m}{\delta_m \beta}\right)$. Also note that $q_{2,2}^s$ is the unique zero of $f_{2,2}(q,\mu)$ in $(0,1)$ (this is straight forward for $\mu \geq 0.5$ and can be shown numerically for $\mu < 0.5$).

Stability of $q_{2,2}^s$ follows from similar arguments as in (i). ■