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Buying Shares and/or Votes for Corporate Control

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Abstract

We revisit questions concerning the implications of voting rights for the efficiency of corporate control contests. Our basic set-up and the nature of the questions continue the work of Grossman and Hart (1988), Harris and Raviv (1988) and Blair, Golbe, and Gerard (1989). We focus on the effect on efficiency of allowing votes to be traded separately of shares in three cases. In addition to outright offers for shares (and for votes when such offers are permitted) we allow the parties competing for control rights to make either offers contingent on winning or quantity restricted offers. Our main conclusion characterizes when allowing vote buying is harmful, and such situations exist for efficiency in all the cases. Allowing quantity restricted offers is also harmful to efficiency (whether or not vote buying is allowed). However, allowing conditional offers is not in itself detrimental to efficiency. These sharp observations are no longer true if we look at the payoff to the initial shareholders alone (ignoring of the benefits of control). In particular, there are parameters for which allowing separate vote trading increases shareholder profits, despite being harmful for efficiency.

The paper also makes a methodological contribution to the analysis of takeover games with a continuum of shareholders. It suggests a way of dealing with the mixed strategies that are crucial for the analysis, develops arguments that facilitate characterization results without fully constructing the set of equilibria and deals fully with the question of existence.

1 Introduction

We study contests over the control of a firm with widely dispersed ownership. The focus is on the implications of allowing the sale of votes separately from shares. There is a substantial recent literature arguing that vote buying occurs in practice (albeit indirectly) but we have not discovered a model that fully characterizes and contrasts the equilibrium outcomes with and without vote trading, and that pinpoints the effect on efficiency and on shareholder profits of allowing for separate vote buying.

This paper is a direct follow up on the early literature on the allocation of voting rights to shares which goes back to Grossman and Hart (1988), Harris and Raviv (1988) and Blair, Golbe, and Gerard (1989). While our basic set-up and the nature of the questions follow this literature, the results obtained are new. A more detailed discussion of the relation to the literature is presented in Section 2 below.

Following the literature, our model features two contestants competing for control—an incumbent and a rival. The rival moves first and makes a tender offer to the shareholders. The incumbent responds with a competing offer. Then the shareholders simultaneously make their tendering decisions that determine which contestant obtains control. The firm generates income for its shareholders and a private benefit for the party in control; the magnitudes of the income and benefits depend on the identity of the parties. In addition to outright offers for shares (and for votes when such offers are permitted) we allow the contestants to make either conditional offers (contingent on winning) or restricted offers (placing a cap on the quantity of shares that will be purchased at the announced price).[1] Our main conclusion is that allowing vote buying is harmful in terms of efficiency in all cases (whether or not quantity restrictions are allowed and whether or not conditional offers are allowed). Allowing restricted offers is also harmful to efficiency (whether or not vote buying is allowed). However, allowing conditional offers is not in itself detrimental to efficiency. There are of course other considerations (e.g., the presence of taxation (Blaire et. al. (1989)) that might associate vote buying with a favorable effect on efficiency. The present work only highlights the costs directly resulting from the forms of contracts allowed. The sharp observations we obtain regarding efficiency no longer hold if we look at the profits of the initial shareholders alone

[1] We assume small shareholders to rule out equilibria where they are pivotal, and assume that the competing parties must make identical offers to all shareholders.

(ignoring the benefits of control). In particular, there are parameters for which allowing separate vote trading increases shareholder profits, despite being harmful for efficiency.

Besides the substantive insights outlined above, the paper also has a methodological contribution to the analysis of takeover games with a continuum of shareholders. It suggests a way of dealing with the mixed or asymmetric strategies that are crucial for the analysis, develops arguments that facilitate characterization results without fully constructing the set of equilibria and deals with the question of existence. Thus, this contribution provides a fully developed model that can be used to study these and related issues.

The original motivation for our interest was to understand the difference between the acquisition of control in the corporate context and vote buying in elections in the political context. Intuitive discussions tend to view the former activity as efficiency enhancing and the latter as detrimental and it is interesting to understand whether and in what sense this might be true. This question has already been discussed to some extent by Dekel, Jackson and Wolinsky (2008). The present analysis deepens the understanding by emphasizing that, in the corporate arena, the acquisition of control could be associated with efficiency only because shares are traded with the votes. Vote buying alone is not efficient in the corporate context as well. In the political arena there is no natural analog to the trading of shares. Such an analog would require that each vote-buying party will receive from (or compensate) the voters who tender their votes to that party any future benefit (or loss) that those voters enjoy (or suffer) from the policies implemented by the winning party. Our analysis does imply that when there are such conditional ex-post transfers allowing vote buying would be efficiency enhancing.

2 Related Literature

As noted the outline of the model and the questions we study are direct continuations of Grossman and Hart (1988; henceforth GH), Harris and Raviv (1988; henceforth HR) and Blair, Golbe, and Gerard (1989; henceforth BGG). Both GH and HR considered situations in which different classes of shares can have different voting rights and argued that, at least in certain regions of the parameter space, one-share one-vote (1S1V) is an "optimal" structure. In some sense, our model considers two classes of shares–ordinary shares and pure voting shares. So, the spirit of their results would be perhaps translated to an argument against allowing the separation of votes from shares. BGG is more closely related in that it also considers directly the possibility of separating votes from shares. They argue that, when offers for shares and/or votes can be made contingent on winning, allowing to trade votes separately from shares does *not* impose any efficiency losses.

All these papers share the same basic structure of tender offers made in sequence to a large population of small shareholders who then make their tendering decisions simultaneously. While GH and HR end up with seemingly the same conclusion, justifying the one-share-one-vote arrangement, they are in fact quite different. First, GH define optimality in terms of the ultimate payoffs to the initial shareholders, while HR define efficiency in terms of the sum of the public value of the firm and the private control benefit, so their conclusions differ. Second, HR's result is driven by pivot considerations of shareholders, while GH's model precludes such considerations by assumption. So, the underlying environments are different. We think that pivot considerations are relevant in a situation in which a small number of large shareholders are holding indivisible blocks of shares, whereas ignoring them seems more suitable for a situation in which the shares are widely distributed among many small shareholders.

We thus follow GH in assuming away pivot considerations. We focus on the efficiency notion of HR, but we also provide some results which use GH's notion. One contribution of our paper to the literature is that we provide a complete equilibrium analysis. This entails the construction of the "mixed" equilibrium in a tendering game with a continuum of agents, the precise identification of the refinements needed to prune away problematic equilibria and the full arguments needed to establish existence. The equilibrium analysis in GH is not complete (it cannot be without the "mixed" equilibrium that we construct) and covers in detail only a certain region of the possible parameter configurations. The choice of model and refinements has also enabled us to generate some sharp and intuitive insights into the effect of vote trading, and of quantity and conditional restrictions. A second contribution is in the substantive results: efficiency obtains without vote buying and quantity restrictions (without the pivotal considerations required in HR's model), and identifying when inefficiency is caused by allowing separate vote trading.

BGG uses the same efficiency notion we do and essentially the same basic model, but reaches the very different conclusion that with *only* contingent offers vote trading does not harm efficiency. This is a meaningful conclusion since in the presence of other elements like taxation it might imply that it is beneficial to allow of vote trading. Our analysis shows that this result does not hold in the natural environment where contenders can make non-contingent offers as well.

Bagnoli and Lipman (1989, henceforth BL) analyze a model in which a raider makes a takeover bid (that is not met by an incumbent's response). They develop a model with a finite number of shareholders and study its limit as the number grows. They contrast this with Grossman and Hart (1980) who analyze the same situation using a model with atomless and non-pivotal shareholders. BL do not define the mixed strategy equilibria of the limit continuum game and hence they do not characterize nor study it directly as we do. Substantively, BL follow Grossman and Hart (1980) in inquiring how the free rider problem might impede takeover attempts. Our substantive focus is instead on the effect of allowing trading of votes separately from shares.

Hirshleifer and Titman (1990) develop a variant of Grossman and Hart (1980), based more on Shleifer and Vishny (1986), wherein the raider has private information and a block of shares (and the incumbent cannot respond to the raiders offer). Hirshleifer and Titman use mixed-strategy equilibria in a manner similar to what we do here to fully solve that model.

3 The Model and Analysis

3.1 The model

This is a model of a contest for control of a firm. Initially, the firm is controlled by the incumbent management team, I, and the shares of the firm are spread uniformly across a continuum of identical shareholders denoted by the interval $[0, 1]$. Each share is bundled with a vote. A rival management team, R, is trying to gain control of the firm by acquiring from the shareholders the majority of the votes. We will refer to R and I as the contenders.

Under R's control the firm has value $w_R > 0$, which is the total value of the income accruing to the shareholders, and R has private benefit $b_R > 0$. Similarly, w_I and b_I represent the firm value and private control benefit under I's control. Thus, if in the end contender k owns a fraction α of the shares after having paid to shareholders the total sum of t, then **contender k's payoff** is $\alpha w_k - t + b_k$ if it wins; and it is $\alpha w_j - t$ if $j \neq k$ wins. When k

wins, the **payoff to a shareholder** who was paid z is $z + w_k$ if this shareholder still owns the share, and just z if not.

To economize a bit on the taxonomy, we assume that $w_I + jb_I \neq w_R + j'b_R$, for any $j, j' \in \{0, 1, 2\}$. This implies in particular that in all scenarios the total value is always maximized under the control of a unique contender.

We consider two basic situations with respect to the allowable trades: one where shareholders can tender only shares (bundled with the votes), and one where shareholders may also sell the votes separately (while keeping the shares and hence the income accruing to them). In the former each contender $k \in \{I, R\}$ quotes a price p_k^s per share; in the latter each quotes a pair of prices (p_k^s, p_k^v) for full shares (including votes) and for just votes (with no claim to income) respectively. In each of these situations, we consider three scenarios that differ in terms of the additional conditions that the contenders may attach to the price offers. In the basic scenario, the contenders are allowed to make only unrestricted price offers: all the shares tendered to them must be purchased at the quoted prices. In other scenarios the contenders are allowed to qualify their price offers with quantity restrictions and conditions. We will present the details of those scenarios later on when we turn to analyze them. Since the basic model is common to all scenarios, we continue to outline the model using the general term "offer" to represent the combination of prices and whatever additional conditions that may accompany them in the different scenarios. Let F_k denote the set of feasible offers, and $f_k \in F_k$ denote an individual offer, by contender $k \in \{I, R\}$.

The contenders move in sequence. First, R makes an offer $f_R \in F_R$ to all shareholders. Then I responds with an offer $f_I \in F_I$ to all shareholders. After observing both offers, shareholders make their tendering decisions simultaneously. Finally, R gains control if following the tendering stage it ends up controlling more than 50% of the votes. Otherwise I remains in control. In other words, the status quo is for I to remain in control unless R obtains more votes than I.

Strategies are defined in the usual way. A strategy σ_R for R, is a feasible offer, $\sigma_R \in F_R$; a strategy σ_I for I prescribes a feasible offer as a function of R's offer, $\sigma_I : F_R \to F_I$; a strategy for a shareholder specifies a tendering decision (whether and which of the offered tendering options to accept) as a function of the offers (f_R, f_I) made by R and I.

A **tendering outcome** is a four-tuple $m = (m_R^s, m_R^v, m_I^s, m_I^v)$ where m_k^h is the fraction of all shares ($h = s$) or votes ($h = v$) that is being tendered to contender $k = R$ or I. (When only shares can be traded $m_k^v \equiv 0$ and we can write (m_R^s, m_I^s) instead.) The tendering outcome fully determines the fraction of votes that each of the contenders end up controlling (e.g., in a scenario in which a contender must purchase all shares and votes tendered to it, R ends up controlling $m_R^s + m_R^v$ of the votes).

We denote by π **the probability that** R **wins**. The set of π's that are compatible with m is denoted by $\Pi(m)$. That is, if $m_R^s + m_R^v > 1/2$ then $\Pi(m) = \{1\}$, if $m_R^s + m_R^v < 1/2$ then $\Pi(m) = \{0\}$, and if $m_R^s + m_R^v = 1/2$ then $\Pi(m) = [0,1]$.[2]

An **outcome** of the tendering subgame following offers f_R and f_I is a pair $(m,\pi)_{f_R,f_I}$ with $\pi \in \Pi(m)$.

3.2 The solution concept

3.2.1 Subgame Perfect Equilibrium

An **equilibrium in the tendering subgame** is an outcome $(m,\pi)_{f_R,f_I}$ satisfying the following: (i) If $m_k^h > 0$, for $h = s$ or v and $k = I$ or R, then shareholders' expected payoff from tendering instrument h to contender k is at least as high as with any other available option. (ii) If some agent does not tender shares nor votes, i.e., $\sum m_k^h < 1$, then shareholders' expected payoff from not tendering is at least as high as with any other available option.

For example, when only shares are traded part (i) implies

$$m_R^s > 0 \Rightarrow p_R^s \geq \max\{p_I^s, \pi w_R + (1-\pi) w_I\},$$

while part (ii) means

$$p_R^s \geq \max\{p_I^s, \pi w_R + (1-\pi) w_I\}.$$

We emphasize that π is determined in equilibrium: π enters the optimality conditions for shareholders, and π must also be consistent with shareholder behavior ($\pi \in \Pi(m)$).

A **SPE in the entire game** given possible offers in F_R, F_I consists of strategies σ_k, $k = R$, I and for each pair of offers f_R, f_I a selection of an equilibrium outcome in the

[2] Letting any π be feasible when $m_R^s + m_R^v = 1/2$ will be necessary for the existence of equilibrium in the tendering subgame.

Note that $m_R^s + m_R^v = 1/2$ and any π can arise as the limit behavior as $N \to \infty$ over a sequence of models with N shareholders who tender to R with an appropriately chosen probability that tends to $1/2$ while the winning probability it implies tends to π.

tendering subgame $(m,\pi)_{f_R,f_I}$ such that neither R nor I can increase the payoff it gets in the resulting outcome $(m,\pi)_{\sigma_R,\sigma_I(\sigma_R)}$ by deviating from its σ_k.

3.2.2 Our solution concept - a refinement of SPE

Our solution concepts refines SPE by imposing two additional requirements. One rules out knife-edge equilibria which rely on shareholder indifference and would not survive perturbations of the game. The other essentially rules out equilibria in the subgame that are Pareto dominated for the shareholders. The formalization of these requirements is as follows.

Definition 1 *The offers f_R, f_I are said to be **tie-free** if $p_k^h \neq p_j^h$ and $p_k^s \neq p_j^v + w_j$ for $h \in \{s,v\}$ and $j \neq k \in \{R,I\}$.*

Definition 2 *A SPE $\left(f_R^*, \sigma_I^*, \left\{(m^*,\pi^*)_{f_R,f_I} : (f_R,f_I) \in F_R \times F_I\right\}\right)$ is **robust** if for any f_R, f_I, and $\varepsilon > 0$, there are tie-free offers $(f_R^\varepsilon, f_I^\varepsilon)$ in an ε-neighborhood of f_R, f_I and an equilibrium in the tendering subgame following $(f_R^\varepsilon, f_I^\varepsilon)$, denoted $(m,\pi)_{f_R^\varepsilon,f_I^\varepsilon}$, such that*

1. $\left|(m^*,\pi^*)_{f_R,f_I} - (m,\pi)_{f_R^\varepsilon,f_I^\varepsilon}\right| < \varepsilon$ *and*
2. $(m,\pi)_{f_R^\varepsilon,f_I^\varepsilon}$ *is not Pareto dominated for the shareholders by any strict equilibrium in the tendering subgame following $f_R^\varepsilon, f_I^\varepsilon$.*

To understand our motivation for (2) note that, as is common in voting games, inefficiencies in our model can arise due to coordination failures. Since our purpose is to focus on the inefficiencies due to the trading rules – in particular whether votes can be sold separately, we adopt a refinement that rules out inefficiencies arising due to coordination failures.

Part (1) of the robustness refinement pins down how ties are broken.[3] In its absence, tie breaking will not be pinned down uniquely by the equilibrium. For example, consider the scenario in which the contenders may only buy shares at unrestricted prices. Consider a subgame after R offers a price $p_R^s \in (w_I + b_I, w_I + 2b_I)$. If I were to offer $p_I^s = p_R^s$ then shareholders would be indifferent between tendering to I or to R and I would profit from this if a bit more than 50% of the shareholders would tender to it, but I would suffer losses

[3] The definition of tie-free offers is stated here only in terms of uncontingent prices p_k^s and p_j^vsince we have not yet introduced the notation for contingent offers. But it will apply to them in the same way as we will note again after introducing the required notation in section 6.

if all shareholders would tender to it. Thus, in this subgame, there are multiple equilibria that differ in how shareholders break ties when they are indifferent. This observation distinguishes this model from some other Bertrand-style models in which tie breaking is uniquely determined in equilibrium. The robustness requirement rules out equilibria of the form just mentioned that are clearly knife-edge. It implies, for example, that in the equilibrium in the subgame following the offers $p_I^s = p_R^s$, the shareholders will not tender both to R and to I.

Henceforth, when we refer to an **equilibrium** of the game we mean a **robust SPE** (except of course when we explicitly refer to SPE or to (Nash) equilibria of the tendering subgame).

3.3 Overview of the analysis

The analysis focuses on the contrast between the case where votes can be traded separately and the case where they cannot. As mentioned above, this comparison is conducted in three different scenarios with respect to the nature of the offers that the contenders may make. The structure of all the cases however is similar and goes as follows.

Section 8 establishes that in all scenarios there exists an equilibrium. In every case we show that there cannot be an equilibrium in which $\pi \in (0, 1)$. The conclusion from these two observations is that, in equilibrium, one of the contenders wins with certainty ($\pi = 1$ or $\pi = 0$). It is then relatively straightforward to rule out one of these possibilities, thereby identifying the equilibrium winner for each configuration of the parameters.

This allows us to draw conclusions regarding the overall efficiency of the equilibrium. By our definition, the outcome is **efficient** if the contender that generates the maximal total value, $w_k + b_k$, wins. This definition is possibly controversial since the private benefit might be viewed as illicit gains. We therefore also use these observations, combined with some properties of the contenders' best replies, to comment on the payoffs that shareholders receive in equilibrium.

Throughout the analysis we stick to the basic scenario outlined above where R must gain control over at least 50% of the votes in order to win. In the appendix we also present results for an alternative scenario in which the contest ends with a vote. Allowing for voting at the end changes the game, because then R does not need to purchase a majority of the votes to obtain control, it is enough that R obtains a majority in the vote at the end. However, the main results are unchanged.

Despite the similarity in the general structures of the proofs, every scenario requires some specialized work, so it is not possible to provide a unified proof. Still to help the reading, we present in the body of the paper only the proofs of the first (and simplest) scenario. The proofs for the remaining cases are relegated to the appendix.

4 Unrestricted and unconditional offers

In this section we consider the simplest trading rule. The contenders' price offers cannot be quantity constrained—they must purchase the entire quantities tendered to them at the prices they quote. The main results of this section are that, when votes cannot be traded separately, the equilibrium outcome is efficient (maximizes $w_k + b_k$), and with vote trading it need not be efficient. We characterize precisely when inefficiency arises if vote trading is allowed. Roughly speaking, the "wrong" contender can win when its private benefits are sufficiently larger than those of the other contender; vote trading enables it to win even when it is not efficient.

4.1 Only shares

In this subsection votes are inseparable from shares. So, a feasible offer by contender $k = R, I$ is a price p_k^s at which it must purchase all shares tendered to it. To gain control R must purchase at least 50% of the shares.

Theorem 1 *The contender with the higher total value, $w_j + b_j$, wins in all equilibria.*

Proof. Follows from the following two lemmas. ■

Lemma 1 *There is no equilibrium in which both contenders win with strictly positive probability, i.e., there is no equilibrium with $\pi \in (0, 1)$.*

Proof. Robustness implies that, in any equilibrium, it cannot be that some shareholders sell some shares to I and some to R because any tie-free offers near (p_R^s, p_I^s) will break the indifference and change the outcome discontinuously. So, if $\pi \in (0, 1)$ arises at equilibrium, it must be that half the shareholders tender to R and half do not tender at all. Hence $p_I^s \leq p_R^s$, and those who do not tender to R hold out to get the expected value $\pi w_R + (1-\pi)w_I$. In such a case

$$p_R^s = \pi w_R + (1-\pi)w_I, \tag{1}$$

for otherwise either all shareholders would tender to R or not at all. Finally, it also must be that $w_I \le p_R^s$ since if $w_I > p_R^s$ this equilibrium would fail the Pareto part of robustness since its outcome (and any sufficiently close outcome) would be dominated by a strict equilibrium in the tendering subgame in which shareholders do not tender at all.

Let u_j denote the profit of $j = I, R$ in the equilibrium with $\pi \in (0,1)$.

$$u_I = (1-\pi)b_I \tag{2}$$

$$\begin{aligned} u_R &= \frac{1}{2}[-p_R^s + \pi w_R + (1-\pi)w_I] + \pi b_R \\ &= \pi b_R \text{ (by (1))} \end{aligned} \tag{3}$$

1. Suppose $w_I + b_I > w_R + b_R$.

   Consider an equilibrium in which $\pi \in (0,1)$. Let $\hat{u}_I$ denote I's profit after offering p_I^s just above p_R^s. Since $p_I^s > p_R^s \ge w_I$ all shareholders will tender to I. Choosing p_I^s in the interval $(p_R^s, p_R^s + \pi[w_I + b_I - w_R - b_R])$ we get

$$\begin{aligned} \hat{u}_I &= -p_I^s + w_I + b_I \\ &> -p_R^s + \pi(w_R + b_R) + (1-\pi)(w_I + b_I) \\ &= \pi b_R + (1-\pi)b_I \ge (1-\pi)b_I = u_I \end{aligned} \tag{4}$$

   where the second equality holds by the equilibrium condition (1). Thus, I can deviate profitably from the putative equilibrium with $\pi \in (0,1)$.

2. Suppose $w_I + b_I < w_R + b_R$.

   Since this is an equilibrium, I cannot profitably outbid R with p_I^s just above p_R^s. That is,

$$\begin{aligned} u_I &\ge b_I + w_I - p_R^s && (5) \\ &= b_I + w_I - [\pi w_R + (1-\pi)w_I] \\ &= (1-\pi)b_I + \pi(w_I + b_I - w_R) && (6) \end{aligned}$$

where the first equality follows from (1). If $w_I + b_I > w_R$, then $u_I > (1-\pi)b_I$ in contradiction to (2). If $w_I + b_I < w_R$, then $\pi \in (0,1)$ may not arise in equilibrium, since $p_R^{s\prime} = w_R$ would guarantee R a win with profit $b_R > \pi b_R = u_R$ in contradiction to the equilibrium hypothesis.[4]

■

Lemma 2 *If $b_I + w_I < b_R + w_R$ then $\pi = 0$ cannot occur in equilibrium; if $b_I + w_I > b_R + w_R$ then $\pi = 1$ cannot occur in equilibrium.*

Proof. Suppose first that $w_I + b_I > w_R + b_R$. It cannot be that $\pi = 1$. If $p_R^s > w_R + b_R$ and $\pi = 1$ then all shareholders tender to R and R has a loss. So, since R's profitability implies $p_R^s \leq w_R + b_R$, I can win profitably with p_I^s just above $w_R + b_R$. Suppose next that $w_I + b_I < w_R + b_R$. If $b_R > 0$ then $p_R^s > \max\{w_I + b_I, w_R\}$, would guarantee profitable win for R, which I can defeat only at a loss, while if $b_R = 0$ then $p_R^s \in (w_I + b_I, w_R)$ (which is a non-empty interval) guarantees a profitable win for R which I can defeat only at a loss. ■

In terms of shareholder payments the equilibrium outcome is not necessarily unique. If $w_I + b_I > w_R + b_R$, then I always wins but there are multiple equilibria as R's behavior can impact payoffs to I and to shareholders. Specifically, depending on R's initial move, shareholder payoffs could range anywhere in $[w_I, w_I + b_I]$. (However, equilibria with payoffs above $w_R + b_R$ involve weakly dominated offers by R.) If $w_I + b_I < w_R + b_R$ then shareholders payoffs are $\max\{w_I + b_I, w_R\}$.

4.2 Both votes and shares

In this scenario votes can be traded separately from shares. The contenders' offers take the form (p_j^s, p_j^v), where p_j^s is the price for the full share (including its vote) and p_j^v is the price per vote offered by $j = R, I$. As above, contenders are committed to purchase any quantities tendered to them.

In this case vote trading interferes with efficiency: the winner is not always the efficient contender (the maximizer of $w_j + b_j$). To gain some intuition, consider the equilibrium when

[4] This argument would fail if $b_R = 0$. In that case there are multiple equilibria, where R can offer any price $p_R^s \in [w_I + b_I, w_R]$ and win with probability $\pi \in [b_I/(w_R - w_I), 1]$ and in all these equilibria R obtains 0 profits.

trade in votes alone is not allowed. Recall that when $w_R + b_R > w_I + b_I$, R wins with $p^s_R = w_I + b_I$ even if $b_I > b_R$. When votes can be traded, R cannot win profitably with any p^s_R (i.e., with any $p^s_R \leq w_R + b_R$).

To see this suppose $w_I + b_I < w_R + b_R < w_I + 2b_I$ and $b_I > b_R$. Consider $p^s_R = w_R + b_R$. If I responds with $p^v_I = w_R + b_R - w_I - \varepsilon$ (which satisfies $p^v_I > b_I > b_R$ for small ε), then there is not an equilibrium in the subgame in which R wins with probability 1, since then selling the vote to I would yield (for small enough ε) $w_R + p^v_I > w_R + b_R = p^s_R$. Contender I winning with probability 1 is also not an equilibrium, since $w_I + p^v_I < w_R + b_R = p^s_R$. Therefore, the only equilibrium is "mixed" with half the shareholders selling shares to R, the other half selling votes to I and $p^s_R = \pi w_R + (1-\pi)\, w_I + p^v_I$. This implies $1 - \pi = (w_R - w_I - \varepsilon)/(w_R - w_I) \approx 1$ and

$$I\text{'s profit} = (1-\pi)b_I - p^v_I/2 \approx b_I - (w_R + b_R - w_I)/2 > 0,$$

confirming that the response by I of $p^v_I = p^s_R - w_I - \varepsilon$ is profitable. Therefore, R cannot win by making a preemptive offer for shares as in the previous case. Of course, this argument does not establish that R cannot win. It only shows that this sort of equilibrium does not survive when votes can be traded. But it does illustrate the general effect of vote trading.

Theorem 2 *The efficient contender wins in equilibrium except in the following regions of the parameter space.*

1. *If $w_I + b_I > w_R + b_R$ and $b_R > 2b_I$, then R wins though I is the efficient contender.*

2. *If $w_I + b_I < w_R + b_R < w_I + 2b_I$ and $b_I > b_R$, then I wins though R is the efficient contender.*

The proof is in the appendix. The method is as before. It is first shown that there are no mixed equilibria in which both contenders win with positive probability. Then for each region of the parameter space one of the contenders is eliminated as a possible winner, which leaves the other as the sole candidate for winning. Since existence is assured, this characterization implies the result.

The detrimental effect of vote trading on efficiency is not entirely surprising. If we think for a moment about a situation in which only votes can be traded,[5] then the party j with

[5] Such a situation does not seem very relevant for the discussion of corporate control, but it might be relevant for other scenarios like political competition.

the higher b_j will win and the outcome obviously is not always efficient. When both shares and votes can be traded this effect does not completely disappear.

4.2.1 First and second mover advantages

The characterization in Theorem 2 reflects both a first-mover and a second-mover advantage.

- **Second mover advantage:** When $w_R + b_R$ is not too much larger than $w_I + b_I$, then I can win with even a small advantage in private benefits, $b_I > b_R$. By contrast if $w_I + b_I > w_R + b_R$ and R's advantage in private benefits is not too large, $b_I < b_R < 2b_I$, then I wins. So, those situations exhibit a second-mover advantage.

  The source of the second-mover advantage is in the ability to make an offer that induces a mixed equilibrium in the tendering subgame in which the second mover acquires just half the shares or votes. This enables the second mover to offer a premium above the true value. The first mover cannot do so for fear of having to pay the premium to all shareholders. So, the second mover can effectively mimic the effect of a quantity restriction even when it cannot be explicitly imposed.

- **First mover advantage:** When $b_R > 2b_I$, R wins regardless of how much greater is $w_I + b_I$ relative to $w_R + b_R$. In contrast, when $b_I > 2b_R$, then I would still lose if $w_R + b_R > w_I + 2b_I$. So, in those situations there is a first mover advantage.

  The source of the advantage is R's ability to make a preemptive offer to buy votes. Even when w_I is far greater than w_R beating such a preemptive offer would result in a loss for I. The fact that such a response would result in a loss for R as well does not help I since R's offer is already in place. For R's preemptive offer to be successful b_R must be more than twice b_I. This is because I can again use its second-mover ability to induce a mixed equilibrium in which it buys only half the shares and hence can offer premium of up to $2b_I$ over their public value.

  More specifically, if $b_R > 2b_I$, and $w_I + b_I > w_R + b_R$, then I cannot win profitably following an initial offer by R of $p_R^v = 2b_I + \varepsilon$. Obviously, I cannot win profitably with $p_I^v \geq p_R^v$. Consider then I's possible responses with p_I^s. If $p_I^s < w_R + 2b_I + \varepsilon$, then all shareholders sell to R so I will lose. If $p_I^s \in [w_R + 2b_I + \varepsilon, w_I + b_I)$ then since $w_I > w_R$ (which follows from $b_R > 2b_I$ and $w_I + b_I > w_R + b_R$), in the equilibrium of the ensuing subgame I cannot win with probability 1. (This is because, if I wins at $p_I^s < w_I + b_I$, then an individual shareholder does better selling to R and earning $w_I + 2b_I + \varepsilon$.) Thus either R wins or it is a mixed equilibrium in which half sell to R and half to I. The latter requires indifference, $p_I^s = \pi w_R + (1-\pi) w_I + p_R^v$, and then I's profits are $(1-\pi) b_I + ((\pi w_R + (1-\pi) w_I) - p_I^s)/2 = (1-\pi) b_I - b_I < 0$ (the expected benefit of control plus the loss on the shares acquired by I which are half of the total).

In contrast, when $b_I > 2b_R$ and $w_R + b_R > w_I + 2b_I$, I cannot win profitably. In this case R can offer to buy shares at $p_R^s = w_I + 2b_I + \varepsilon$ against which I has no profitable response. Again it is obvious that no offer p_I^s for shares can be beneficial to I. An offer with $p_I^v < 2b_I$ attracts no shareholders, while an offer of $p_I^v > 2b_I$ induces an equilibrium in the subgame with shareholders tendering to both in which I's profit is negative: $(1-\pi) b_I - p_I^v/2 < 0$.

4.2.2 Shareholder profits

The total surplus, $w_j + b_j$, is a potentially controversial measure of social welfare, since the private benefit b_j might represent at least in part illicit gains. We therefore also examine the effect of vote trading on shareholders' payoffs. The comparison of payoffs across the different regimes is sometimes ambiguous due to the presence of multiple equilibria: when I wins in equilibrium, the payoffs to I and to the shareholders depend on R's initial actions, and R is indifferent among a wide range of actions. However, even when the comparison is unambiguous, it can go either way: the introduction of separate vote trading sometimes enhances and sometimes harms shareholders payoffs.

For example, when $w_I + b_I > w_R > w_I$ and $b_R > \min\{w_I - w_R + 2b_I, b_I\}$ contender R wins whether or not votes can be traded separately, but shareholders payoffs with vote trading ($\min\{w_I + 2b_I, w_R + b_I\}$) are larger than without it ($w_I + b_I$). The intuition behind this observation is that vote trading benefits the shareholders because it forces R to make a more aggressive offer. When votes cannot be traded, for R to win it must offer $p_R^s = w_I + b_I$. When votes can be traded, if R simply offers $p_R^s = w_I + b_I$, then I can respond with $p_I^v = b_I - \varepsilon$ and, for sufficiently small ε, will win profitably with probability close to 1 (the equilibrium in the tendering subgame following these offers is mixed). Therefore, R must either offer $p_R^s = w_I + 2b_I$ or $p_R^v = b_I$ to deter I, both of which lead to higher payoffs to shareholders.

By contrast, when $w_R < w_I$ and $b_R > w_I + b_I - w_R > 2b_I$, contender R wins whether or not votes can be traded separately, but shareholders payoffs with vote trading ($w_R + 2b_I$) are smaller than without it ($w_I + b_I$). This is because in the absence of vote trading R has to offer $p_R^s = w_I + b_I$, while with vote trading it can win with buying just votes at $p_I^v = 2b_I$.

Thus vote trading can benefit shareholders because it may force R to make a more aggressive initial offer when faced with the possibility of subsequent offers for votes. It can be harmful under other parameters because R may win control by buying only votes at a lower price than if R had to buy shares.

5 Restricted offers

The change from the previous analysis is that the contenders are allowed to make restricted offers that cap the quantities of shares and/or votes that they will buy at the prices they announce. That is, a contender is committed to buy at the price it announced any quantity tendered to it up to the pre-announced quota. Intuitively, it seems that such a cap should enable contenders to offer higher premiums over the public value of the shares, since by capping the quantity they would not have to pay this premium to all shareholders. It therefore should bias the outcome in favor of contenders with higher private benefits. This intuition is indeed confirmed by the following analysis.

5.1 Only shares

First consider the case in which votes can be transferred only by trading shares. As before, the rival has to acquire a majority of the shares to take control. An offer f_j by contender $j = R, I$ is a pair $f_j = (p_j^s, \overline{m}_j^s)$. This is a commitment to buy at the price p_j^s any quantity tendered to it up to $\overline{m}_j^s$. Recall that the outcome of the ensuing tendering subgame is $(m_R^s, m_I^s; \pi)$ where m_j^s is the mass of shareholders who decide to tender to $j = R, I$ and π is the probability that R wins. If $\overline{m}_j^s < m_j^s$, then the m_j^s shareholders who tendered to j are rationed with equal probability and only a fraction $\overline{m}_j^s / m_j^s$ end up tendering.

Thus, if (π, m_R^s, m_I^s) is an equilibrium outcome of the tendering subgame it must satisfy the following conditions.

- If $m_j^s > 0$, then tendering to j should be at least as beneficial as the alternative options of tendering to the other bidder or keeping the share. That is,

$$\begin{aligned}
&\min\left\{\frac{\overline{m}_j^s}{m_j^s},1\right\}p_j^s+\left[1-\min\left\{\frac{\overline{m}_j^s}{m_j^s},1\right\}\right]\times[\pi w_R+(1-\pi)w_I] \\
\geq\ & \max\left\{\min\left\{\frac{\overline{m}_j^s}{m_j^s},1\right\}p_{-j}^s+\left[1-\min\left\{\frac{\overline{m}_j^s}{m_j^s},1\right\}\right]\times[\pi w_R+(1-\pi)\,w_I]\,,\right. \\
&\left.\pi w_R+(1-\pi)\,w_I\right\}
\end{aligned}$$

Here $\min\left\{\left(\overline{m}_j^s/m_j^s\right),1\right\}$ is the proportion of shareholders who offer their shares to j and succeed in selling them. These shareholders obtain p_j^s while the others receive $\pi w_R+(1-\pi)\,w_I$. The max is over the option of offering one's share to $-j$ and not tendering at all.

- If $m_R^s+m_I^s<1$, then the option of not tendering is at least as beneficial as tendering. That is, for each $j=R,I$

$$\pi w_R+(1-\pi)w_I\geq\min\{(\overline{m}_j^s/m_j^s),1\}p_j^s+[1-\min\{(\overline{m}_j^s/m_j^s),1\}][\pi w_R+(1-\pi)w_I]$$

Remark 1 *We specify that if $\pi_R^s=1/2$ and $m_R^s>1/2$ then R wins.*

The main intuition of the following analysis is that, since the winning contender can cap its offer at half the shares, it can bid up to w_j+2b_j and still break even. Therefore, we expect that I wins if $w_I+2b_I>w_R+2b_R$ and R wins if the reverse inequality holds strictly.

Theorem 3 *In all equilibria the contender with the higher value of w_j+2b_j wins.*

The proof is in the appendix and again follows the logic of first ruling out equilibria with $\pi\in(0,1)$.

5.2 Both votes and shares

An offer f_j by $j=R,I$ is a four-tuple $f_j=\left(p_j^s,\overline{m}_j^s;p_j^v,\overline{m}_j^v\right)$, where p_j^s and p_j^v are the prices offered by j for shares and votes respectively, while $\overline{m}_j^s$ and $\overline{m}_j^v$ are the respective quantity restrictions. The main result here is that vote buying harms efficiency in the sense that the region of the parameter space over which the efficient contender wins shrinks in comparison to the case in which votes cannot be traded separately.

An outcome of the tendering subgame following f_R and f_I is $(m,\pi)_{f_R,f_I} = (m_R^s, m_R^v, m_I^s, m_I^v; \pi)_{f_R,f_I}$, where m_j^s and m_j^v are the masses of shareholders who decide to tender shares and votes respectively to $j = R, I$ given offers (f_R, f_I) and as before π is the probability that R wins following these offers. The rationing rules are as before and are applied to each offer separately. If $\overline{m}_j^s < m_j^s$ only a fraction $\overline{m}_j^s/m_j^s$ end up tendering shares to j and similarly if $\overline{m}_j^v < m_j^v$ only a fraction $\overline{m}_j^v/m_j^v$ end up tendering votes to j, independently of contender j's other offer. At such an outcome, the expected payoff of tendering shares to j is $\min\{(\overline{m}_j^s/m_j^s), 1\}p_j^s + [1-\min\{(\overline{m}_j^s/m_j^s), 1\}][\pi w_R + (1-\pi)w_I]$; the expected payoff of tendering votes to j is $\min\{(\overline{m}_j^v/m_j^v), 1\}p_j^v + [\pi w_R + (1-\pi)w_I]$. In an equilibrium of the tendering subgame, any action taken by a positive mass of shareholders (tendering shares and/or votes or not tendering at all) must yield to shareholders expected payoff at least as high as the expected payoff of any of the available options of tendering or not.

Remark 2 *As in remark 1 if R is oversubscribed when it restricts its purchases to half the shares and votes then it wins. That is, if* $\min\{\bar{m}_R^v, m_R^v\} + \min\{\bar{m}_R^s, m_R^s\} = 1/2$ *and* $m_R^s > \bar{m}_R^s$ *or* $m_R^v > \bar{m}_R^v$ *then R wins.*

Theorem 4 *The identity of the winner is the same as in Theorem 3 except for parameter configurations satisfying $w_I + 2b_I > w_R + 2b_R$ and $b_R > b_I$. For these configurations I is the efficient contestant and would be the winner in the absence of vote trading, but R wins when vote trading is allowed.*

The proof is in the appendix and its logic is again as in previous cases. It is argued first that in all equilibria $\pi \notin (0,1)$. Then for each region of the parameter space either $\pi = 0$ or $\pi = 1$ is ruled out which implies (via existence) that the remaining case prevails in equilibrium.

5.2.1 First and second mover advantages

The results above show that with restricted offers there is only a first-mover advantage (and no second-mover advantage). This is consistent with the reason for the second-mover advantage when restricted offers are not possible. There we argued that the second-mover advantage results from the ability of the second mover to create a mixed equilibrium in the tendering subgame in which the second mover obtains half the votes, but that the first mover cannot do so for fear of having to pay all the shareholders. With the ability of making restricted offers this limitation on the first mover does not exist, and the first mover can do exactly what the second mover achieves. Indeed the first mover, R, wins with restricted offers in strictly more cases than he does when he cannot make restricted offers.

6 Contingent offers

In this scenario contenders are allowed to make contingent offers, an offer which takes effect if and only if the offering contender wins. An offer by contender $k = I, R$ for shares is a pair of prices: a contingent price p_k^{sc} at which contender k will buy all shares that were tendered to it in the event that it wins and a non-contingent price p_k^s at which it is committed to buy in any case. Similarly An offer by contender $k = I, R$ for votes specifies a contingent price p_k^{vc} and a non-contingent price p_k^v. Each of these prices stands for a contender's commitment to purchase any quantity tendered subject to the contingency.

Now that we have the notation, we restate Definition 1 of tie-free offers to apply to contingent offers as well: The offers f_R, f_I are tie-free if $p_k^h \neq p_j^h$ and $p_k^s \neq p_j^v + w_h$ for $h \in \{s, v, sc, sv\}$ and $j \neq k \in \{R, I\}$.

6.1 Only Shares

Again we first consider the case in which only shares can be traded. An outcome of the tendering subgame is an array of the form $(m_R^s, m_R^{sc}, m_I^s, m_I^{sc}, \pi)$. Thus, the offers are unrestricted offers but they can be conditioned on winning. The main result here is that outcome is efficient—the contender with the highest $w_k + b_k$ wins—as in the case of non-contingent and unrestricted offers for shares alone. Thus, unlike quantity restrictions this form of contingency does not interfere with efficiency.

Theorem 5 *If $w_k + b_k > w_j + b_j$ then in all equilibria k wins.*

The proof is the appendix and its method is again to rule out mixed equilibria in which both contenders win with positive probability. We know from the analysis in section 4.1 that there is no such equilibrium when both contenders make non-contingent offers. This conclusion is extended here to the cases in which at least one contender makes a conditional offer.

6.2 Both votes and shares

Now allow for votes to be traded separately. Here, an outcome of the tendering subgame is an array of the form $(m_R^s, m_R^{sc}, m_R^v, m_R^{vc}, m_I^s, m_I^{sc}, m_I^v, m_I^{vc}, \pi)$. The analysis is similar to the case with non-contingent, unrestricted offers. While more complicated as there are more cases to consider, surprisingly the outcome is unaffected by allowing for contingent offers.

Theorem 6 *The efficient contender wins in equilibrium except in the following regions of the parameter space.*

1. *If $w_I + b_I > w_R + b_R$ and $b_R > 2b_I$, then R wins.*

2. *If $w_I + b_I < w_R + b_R < w_I + 2b_I$ and $b_I > b_R$, then I wins.*

The proof is in the appendix and the argument follows the same logic of ruling out mixed equilibria as in the previous proofs.

7 Variations on the basic model: voting in the end

In the version of the model analyzed so far, R gains control only if it acquires more than 50% of the votes. In an alternative description of the process the bidding contest is followed by a vote that determines which contender will end up in control. In such a case, R might gain control even when it does not acquire the majority of the votes. It is not entirely clear which is the “right” model. Some related contributions in the finance literature employ the former model and some employ the latter. The rationale for using the model without the voting in the end is that to force a vote on control the raider might have to acquire a majority of the votes.

However, this question is not particularly important for our analysis, since the introduction of voting to the model would not change the results. To see this, consider a modified version of the model with voting in the end. That is, once the tendering stage is over, the two contenders with the blocks they have acquired and the remaining shareholders (who have not sold their vote nor share) vote and the contender who wins this vote gains control. We will establish the claim by showing that any equilibrium outcome in the voting version has an equivalent outcome with the same winning probabilities in the game without voting. We present the argument for the environments in which the contenders can make unrestricted offers for shares or for both shares and votes. It is clear that the argument can be extended to the case of restricted offers as well, but this will require some additional steps and we will forgo it here.

Observe first that, if $w_R < w_I$, those who do not tender to R end up voting for I, so in order to win R must still acquire over 50% of votes and nothing changes in the above analysis. Consider, therefore, the case of $w_R > w_I$ and a particular equilibrium in this case. Let π denote the probability with which R wins, and θ_k denote the fraction of the total votes (with or without shares) that $k = R, I$ ends up purchasing in this equilibrium. Clearly, if $\theta_R > 1/2$, this equilibrium is automatically an equilibrium in the absence of voting as well. Similarly, if $\pi = 0$, this is also the case, since if R cannot deviate profitably when there is voting in the end, it cannot do so in the absence of voting. Finally, if $\pi > 0$ and $\theta_R \leq 1/2$, consider a configuration which differs from the equilibrium configuration only in that R offers an unrestricted price for shares $p_R^s = \pi w_R + (1 - \pi) w_I$ (i.e., the other parts of R's offer and those of I's offer are just as in the equilibrium); all the shareholders who tender shares to R or vote for R in the equilibrium sell shares to R at this p_R^s and all other shareholders behave as in the equilibrium. It can be verified that this configuration is an equilibrium outcome in the game without voting in the end. The shareholders who sell shares to R at p_R^s get the same payoff as those voting for R in the equilibrium and so do the shareholders who sell to I or to another part of R's offer. Both R and I get the same payoffs. Clearly, R does not have a profitable deviation, since it would be available in the equilibrium with voting as well. Similarly any profitable deviation by I would have the same effect in the equilibrium with voting. Thus, the constructed configuration is an equilibrium configuration in the game without voting.

8 Existence

In this section we prove existence of an equilibrium. The method is to consider limits of equilibria of a sequence discretized games (where the actions spaces of I and R are finite, and there is a continuum of shareholders). The grids for the discretized games are selected so as to preclude ties (i.e., in our terminology, any pair of offers in a discretized game is "tie-free").

Recall the notation f_j, $j = I, R$, is an offer, F_j is the set of feasible offers for j, and an outcome in the tendering subgame following (f_R, f_I) is a tuple of the form $(m, \pi)_{f_R, f_I} = (m_R^s, m_R^v, m_I^s, m_I^v; \pi)_{f_R, f_I}$ consisting of the fractions of shareholders tendering shares and votes to each firm, and the probability π with which R wins. Let $C(f_R, f_I)$ denote the set of equilibrium outcomes in the tendering subgame which are not Pareto dominated by a strict equilibrium outcome in the tendering subgame. Let $u_j\left(f_R, f_I, (m, \pi)_{f_R, f_I}\right)$ denote the payoff to contender j given f_R, f_I, and an outcome $(m, \pi)_{f_R, f_I}$ in the subgame following (f_R, f_I). Finally let $U_j(f_R, f_I) = \{u_j(f_R, f_I, (m, \pi)) : (m, \pi) \in C(f_R, f_I)\}$.

F_j varies across the different scenarios as follows.

- In the unrestricted-shares case $F_j = \mathbf{R}_+$ is a set of p^s's (prices for shares)
- In the case of unrestricted shares and votes $F_j = \mathbf{R}_+^2$ is a set of (p^s, p^v) pairs (prices for shares and for votes)
- In the quantity-restricted shares case $F_j = \mathbf{R}_+ \times [0, 1]$ is a set of $(p^s, \overline{m}^s)$ pairs (share price and quantity restriction)
- In the case of quantity-restricted shares and votes $F_j = (\mathbf{R}_+ \times [0, 1])^2$ is a set of $(p^s, \overline{m}^s; p^v, \overline{m}^v)$ 4-tuples (prices and corresponding quantity restrictions)
- In the case of contingent offers for shares $F_j = \mathbf{R}_+^2$ is a pair of prices p^s and p^{sc} (uncontingent or contingent).
- In the case of contingent offers for shares and votes $F_j = \mathbf{R}_+^4$ is a pair of pairs of prices, one pair corresponds to the contingent and uncontingent offers for shares and the other for votes.

First note that C is a non-empty correspondence. This follows from existence of equilibria in the shareholder subgame. Fix the offers, f_R, f_I. For each $\pi \in [0, 1]$, define the set of tendering outcomes $M(\pi)$ that are optimal for the shareholders when they expect R to win with probability π. (That is, given π, if $m_k^h > 0$ then tendering h to k maximizes the shareholder's utility out of the available options, and if $\sum_{k,h} m_k^h < 1$ then not tendering must be optimal.) Clearly this set of tendering outcomes is non-empty, convex valued and the correspondence $M(\pi)$ is upper hemi-continuous. Recall that for each outcome $m \in M(\pi)$

the correspondence $\Pi(m)$ defines the set of π's that are consistent with m. (That is, if R's share of the votes at that outcome is strictly smaller than $1/2$ or strictly larger than $1/2$, then the resulting set is $\{0\}$ or $\{1\}$ respectively; if R's share of the votes is exactly $1/2$ then the resulting set is $[0,1]$.) So $\Pi(M(\pi))$ defines a non-empty, convex valued, upper hemi-continuous correspondence whose fixed point is an equilibrium value of π for the tendering subgame. This implies that the set of all equilibrium outcomes $(m,\pi)_{f_R,f_I}$ in the tendering subgame following (f_R, f_I) is non empty, and obviously $C(f_R, f_I)$ is a non-empty subset.

Now consider a different type of game in which we, the analysts, choose a selection of C. That is, we choose a function c defined on $F_R \times F_I$ such that $c(f_R, f_I) \in C(f_R, f_I)$ and other than that the game is the same as the original game. We call this the new game, and the preceding version – where the shareholders get to choose any equilibrium outcome of the tendering subgame from C – the original game.

Claim 1 *Given a SPE of the original game there is a selection c under which those strategies are a SPE of the new game, and conversely, given a selection c and a SPE equilibrium of the new game, we have a SPE of the original game.*[6]

Proof. Obvious. ■

Remark 3 *In the original game there is no selection from U_i that is continuous. Equivalently, there is no selection c such that the new game is continuous. To see this consider, for example, parameters satisfying $\min_i(w_i + b_i) > p_I^s > p_R^s > \max_i w_i$. Then the only outcome that is not Pareto dominated by a strict equilibrium outcome in the tendering subgame is for all shareholders to sell to I. Consider $p_R^s > p_I^s > \max_i w_i$: then all sell to R. So if we have a sequence converging to $p_I^s = p_R^s$ continuity must fail: whatever we think shareholders do, the game is not continuous.*

Claim 2 *C and U are upper hemi-continuous.*

Proof. Obvious. ■

Remark 4 *Note that if the set C was defined to include only Pareto undominated equilibrium outcomes in the tendering subgame (rather than all those that are undominated by* strict

[6] Here and elsewhere in this section the term SPE refers to any subgame perfect equilibrium not necessarily a robust one (which we refer to as an equilibrium thoughout the paper).

equilibria of the tendering subgame), then we would not obtain upper hemi-continuity. Indeed, consider a game with $w_R > w_I$ and a subgame after $p_R^s = 0, p_R^v < b_I$. Then I has no best reply. I would want to choose $p_I^v = p_R^v$ and sell to all but this will be Pareto dominated for the shareholders by an (non-strict) equilibrium in the subgame in which all sell their votes to R. If I chooses $p_I^v = p_R^v + \varepsilon$ then I gets $u_I^\varepsilon = b_I - p_R^v - \varepsilon$, so I wants to choose $\varepsilon > 0$ as small as possible.

Now define another game, call it an extended game.[7] The extended game has three players. The incumbent and rival have the same strategy space, and a fictitious third player chooses an element of $\mathbf{R}^2$. The payoffs are as follows. I gets whatever the third player chooses for him, R gets whatever the third player chooses for him, the third player gets 1 if the vector of strategies are any element of $\{(f_R, f_I, U_R(f_R, f_I), U_I(f_R, f_I))\} \subset F_R \times F_I \times \mathbf{R}^2$ and is a continuous function that strictly decreases as the strategies move away from that set. The payoffs for I and R are trivially continuous. The payoffs for the third player are continuous if (and only if) both U_k's are upper hemi-continuous.

Claim 3 *A SPE of the extended game is a SPE of a new game (where we use the selection c given by the third player from the extended game), and conversely.*

Proof. Obvious. ■

Claim 4 (Hellwig et. al. (1990)) *Given any sequence of finite grids of a continuous extensive form game, and any sequence of SPE for the sequence of games, the limit of the path of those SPE is a SPE path of the limit game. (Take subsequences whenever necessary.) Moreover, there exists a sequence of SPE of the finite games converging to the SPE of the limit game.*

Proof. The first claim is Theorem 1 in Hellwig et. al. (1990). The second claim follows from their discussion of lower hemi-continuity (p. 419). ■

Our existence result now follows from the above arguments.

Proposition 1 *In each of the scenarios considered in this paper there exists a SPE whose outcome is a limit of SPE outcomes in a sequence of discretized versions of the game converging to the original game.*

[7] We thank Phil Reny for this idea.

Proof. Take a sequence of finite-grid games G_n converging to the original game, and take any convergent sequence of outcomes e_n such that e_n is a SPE of G_n. Any such outcome e_n is also a SPE outcome of an extended version of G_n (by the construction above). Hence, the extended version of the limit game has a SPE and furthermore the sequence e_n converges to the outcome of that SPE (by Hellwig et. al. (1990)). The SPE that supports that outcome in the extended version of the limit game is a SPE of the original game that has the same outcome (by the construction above). ■

We conclude by claiming that (robust) equilibria exists. First we make a trivial observation that follows from the definition of robustness.

Claim 5 *Fix a sequence of grids without ties,* F_k^n*,* $k = R, I$*, such that* $F_k^n \to F_k$*. If* $\left(f_R^n, \sigma_I^n, \left\{(m^n, \pi^n)_{f_R, f_I} \in C(f_R, f_I) : f_R, f_I \in F_R^n \times F_I^n\right\}\right)$ *is a sequence of (robust) equilibria with* $f_R^n \to f_I$*,* $\sigma_I^n \to \sigma_I$ *(i.e., for all* $f_R \in F_R$ *there is a sequence* $f_R^n \to f_R$ *with* $\sigma_I^n(f_R^n) \to \sigma_I(f_I)$*) and* $(m^n, \pi^n) \to (m, \pi)$ *(i.e., for all* (f_R, f_I) *there is a sequence* $(f_R^n, f_I^n) \to (f_R, f_I)$ *with* $(m^n, \pi^n)_{f_R^n, f_I^n} \to (m, \pi)_{f_R, f_I}$*) and* $\left(f_R, \sigma_I, \left\{(m, \pi)_{f_R, f_I} : f_R, f_I \in F_R \times F_I\right\}\right)$ *is a SPE then* $\left(f_R, \sigma_I, \left\{(m, \pi)_{f_R, f_I} : f_R, f_I \in F_R \times F_I\right\}\right)$ *is a (robust) equilibrium.*

Proof. This is just a restatement of the definition of robust equilibrium. ■

Proposition 2 *A robust equilibrium exists in all the games considered in this paper.*

Proof. Follows from Claims 4 and 5 and Proposition 1. ■

Remark 5 *Notice that the set of (robust) equilibrium outcomes is contained in the set of outcomes of SPE that satisfy the tie-free part of the robustness definition and such that, for any offers* (f_R, f_I)*,* $(m, \pi)_{(f_R, f_I)} \in C(f_R, f_I)$*. This because, if an outcome* $(m, \pi)_{(f_R, f_I,)}$ *is not an element of* $C(f_R, f_I)$ *because it is Pareto dominated by a strict equilibrium, say* $(\hat{m}, \hat{\pi})$ *in the tendering subgame, then it will also fail robustness. To see this recall that robustness requires* $(f_R^\varepsilon, f_I^\varepsilon)$ *close to* (f_R, f_I) *and* $(m^\varepsilon, \pi^\varepsilon)$ *an equilibrium in the subgame following* $(f_R^\varepsilon, f_I^\varepsilon)$ *such that* $(m^\varepsilon, \pi^\varepsilon)$ *is not dominated by any strict equilibrium in the subgame following* $(f_R^\varepsilon, f_I^\varepsilon)$*. But for* ε *small enough* $(\hat{m}, \hat{\pi})$ *will be a strict equilibrium in the subgame following* $(f_R^\varepsilon, f_I^\varepsilon)$ *and it will Pareto dominate* $(m^\varepsilon, \pi^\varepsilon)$*. Thus characterization results that hold for all SPE that satisfy this weaker condition, hold automatically for all the (robust) equilibria.*

9 Conclusion

This paper makes two types of contributions. First, it makes a methodological contribution to the analysis of takeover games with a continuum of shareholders. It suggests a way of dealing with the mixed strategies that are crucial for the analysis, develops arguments that facilitate characterization results without fully constructing the set of equilibria and deals with the question of existence. This opens the way both to examine and fully understand the scope of old results and to generate new results. Second, the analysis obtains relatively sharp substantive insights and shows that earlier conclusions might be misleading. The practice of vote buying is detrimental to efficiency under all circumstances, but is not necessarily detrimental to shareholder profits. Thus, previous conclusions about the efficiency of vote buying when contingent offers are allowed and about the optimality of one share-one vote for shareholders payoffs are imprecise or incomplete.

10 Appendix

10.1 Proofs for subsection 4.2

Theorem 2 The efficient contender wins in equilibrium except in the following regions of the parameter space:

1. If $w_I + b_I > w_R + b_R$ and $b_R > 2b_I$, then R wins though I is the efficient contender.
2. If $w_I + b_I < w_R + b_R < w_I + 2b_I$ and $b_I > b_R$, then I wins though R is the efficient contender.

The proof relies on Lemma 3 (which adapts Lemma 1 to this case) and on Propositions 3 and 4 which are stated and proved below. The analysis is simplified by noticing that wlog I need only make an offer for either shares or votes, but not both together. If shareholders sell only votes or only shares then of course the other offer is irrelevant. If shareholders are indifferent and buy both then they must be indifferent so that $p_I^s = \pi w_R + (1-\pi) w_I + p_I^v$, and then I is indifferent as well. This argument does not apply to R as an offer that is not taken in equilibrium may still restrict I's replies.[8]

[8]For example, if $p_R^v + \pi w_R + (1-\pi) w_I = p_I^v + \pi w_R + (1-\pi) w_I = p_R^s$ it may be that no shareholders buy votes from R and I fails to lower p_I^v as that would result in no one selling votes to I. But if R were to

Lemma 3 *There is no equilibrium in which both contenders have a strictly positive probability of winning, i.e., there is no equilibrium with $\pi \in (0,1)$.*

Proof. Note that in any equilibrium with $\pi \in (0,1)$ contender R purchases half the votes (with or without the shares), and the shareholders are indifferent. As in the proof of Lemma 1, robustness implies that, in any equilibrium, it cannot be that some shareholders sell some shares to I and some to R because any tie-free offers near (p_R^s, p_I^s) will break the indifference and change the outcome discontinuously. The proof of Lemma 1 also shows that it cannot it arise due to shareholder indifference between tendering shares to R and not tendering (note that the argument there applies since such indifference requires $p_I^v = p_R^v = 0$.) Therefore, $\pi \in (0,1)$ can arise only in two cases. (1) After (p_R^v, p_R^s, p_I^v) such that $p_R^s \geq \min w_k$, $p_I^v \in (p_R^s - \max w_k, p_R^s - \min w_k)$, and $p_I^v \geq p_R^v$ and no one sells votes to R.[9] (2) After (p_R^v, p_R^s, p_I^s), such that $p_I^s \in (p_R^v + \min w_k, p_R^v + \max w_k)$ and $p_I^s \geq p_R^s$ and no one sells shares to R.[10] Outside the closure of these open intervals R or I wins with certainty since all shareholders prefer selling either to I or to R regardless of π. (At the endpoints of these intervals we have $p_k^s = p_j^v + w_l$ for $j \neq k$ and $l = I$ or R, which precludes $\pi \in (0,1)$ as shareholders indifference requires $p_k^s = p_j^v + \pi w_R + (1-\pi) w_I$ and $w_I \neq w_R$.)

First, consider the tendering subgame after offers $p_R^s \geq \min w_k$ and $p_I^v \geq 0$ such that $p_I^v \in (p_R^s - \max w_k, p_R^s - \min w_k)$.

Assume $w_I > w_R$, so that $p_I^v \in (p_R^s - w_I, p_R^s - w_R)$. The Pareto undomination part of the robustness requirement then selects $\pi = 0$.

Assume $w_I < w_R$ so that $p_I^v \in (p_R^s - w_R, p_R^s - w_I)$. Then, $\pi \in (0,1)$ implies

$$p_R^s = \pi w_R + (1-\pi) w_I + p_I^v \tag{7}$$

and so

$$\pi = \frac{p_R^s - w_I - p_I^v}{w_R - w_I}. \tag{8}$$

hence

lower p_R^v then I could lower p_I^v and not lose all votes.

[9] No one sells votes to R because in any tie-free offers either $p_I^v > p_R^v$ and no sells votes to R or $p_I^v < p_R^v$ and then no one would sell votes or shares to I, and in both events, by the tie-free part of the robustness requirement, π would not be interior.

[10] See footnote 9.

$$
\begin{aligned}
u_I &= (1-\pi)\, b_I - p_I^v/2 \\
&= \frac{w_R - p_R^s + p_I^v}{w_R - w_I} b_I - p_I^v/2
\end{aligned} \tag{9}
$$

Notice that u_I describes the profit at the purported mixed equilibrium. Moreover, for other $p_I^v \in (p_R^s - w_R, p_R^s - w_I)$ this function continues to describe the payoffs to I so long as $p_I^v > p_R^v$.

If $w_I + 2b_I > w_R$, then u_I is increasing in p_I^v so I has a profitable deviation from the purported equilibrium.

If $w_I + 2b_I < w_R$ then u_I is decreasing in p_I^v and if $p_I^v > p_R^v$ then there is again a profitable deviation for I from the purported equilibrium.

Thus, the only possibility for $\pi \in (0,1)$ is that $w_I + 2b_I < w_R$ with $p_R^s \leq w_R$ (since if $p_R^s > w_R$ then $u_I < 0$ by (9)) and $p_I^v = p_R^v$ (and no one sells votes to R). But this is ruled out as follows.

R's payoff at the purported equilibrium is

$$
\begin{aligned}
u_R &= \pi b_R + \frac{\pi w_R + (1-\pi)\, w_I - p_R^s}{2} \\
&= \pi b_R - p_I^v/2 \\
&= \frac{p_R^s - w_I - p_I^v}{w_R - w_I} b_R - p_I^v/2.
\end{aligned} \tag{10}
$$

which is increasing in p_R^s and decreasing in p_I^v. If R deviates to $p_R^s = w_R$ and $p_R^v = 0$ then I will not respond with $p_I^s \geq p_R^s$ (since if the last inequality is strict then $u_I = w_I - p_I^s < w_I - w_R < 0$ and if it is an equality then by the tie-free part of the robustness requirement either I buys from all and also $u_I = w_I - p_I^s = w_I - w_R < 0$ or R buys from all and $u_I = 0$), and as established above in this case u_I is decreasing in p_I^v so I's best response in terms of p_I^v is $p_I^v = 0$. Therefore, the deviation to $p_R^s = w_R$ and $p_R^v = 0$ increases u_R, so R has a profitable deviation unless $p_R^s = w_R$ and $p_R^v = 0$. But then, as noted, I's best reply is $p_I^v = 0$ whereupon $\pi = 1$. This establishes that in the subgame following an offer $p_R^s \geq \min w_k$, there is no equilibrium with $\pi \in (0,1)$

Second, consider the equilibria in the subgame following (p_R^v, p_I^s), such that p_I^s is in the interval $(p_R^v + \min w_k, p_R^v + \max w_k)$.

If $w_R > w_I$ then there are multiple shareholder equilibria, but again the Pareto undomination part of the robustness requirement selects the equilibrium where all sell to R so $\pi = 1$.

If $w_I > w_R$ then shareholder indifference implies

$$p_R^v + \pi w_R + (1-\pi)\, w_I = p_I^s \tag{11}$$

and hence

$$\pi = \frac{p_R^v + w_I - p_I^s}{w_I - w_R}. \tag{12}$$

$$\begin{aligned} u_I &= (1-\pi) b_I + \frac{\pi w_R + (1-\pi)\, w_I - p_I^s}{2} \\ &= \frac{p_I^s - w_R - p_R^v}{w_I - w_R} b_I - \frac{p_R^v}{2} \end{aligned} \tag{13}$$

which is linear and increasing in p_I^s over $[w_R + p_R^v, w_I + p_R^v]$. Therefore $\max u_I$ is achieved at $p_I^s = w_I + p_R^v$, where $\pi = 0$. Thus, if $w_I > w_R$ then $\pi \notin (0,1)$.

It follows that for all parameter configurations $\pi \in (0,1)$ does not arise on the equilibrium path. ■

Proposition 3 *If (i) $w_R + b_R > w_I + 2b_I$ or (ii) $b_R > 2b_I$, or (iii) $w_R + b_R > w_I + b_I$ and $b_R > b_I$, then I may not win in equilibrium.*

Proof. (A) If $w_R + b_R > w_I + 2b_I$ and $w_R > w_I$, then R can start with p_R^s in the interval $(\max\{w_I + 2b_I, w_R\}, w_R + b_R)$ and win profitably. To see this, observe first that it would not be profitable for I to respond with $p_I^s \geq p_R^s > w_I + 2b_I$. Suppose next that I responds with p_I^v. Clearly $p_I^v < p_R^s - w_R$ leads to $\pi = 1$ (this inequality implies that selling shares to R is better for shareholders than selling votes to I) and $p_I^v > p_R^s - w_I$ leads to losses for I (since then $p_I^v > 2b_I$ and the best I can do is buy half the votes and obtain control with probability 1). For $p_I^v \in [p_R^s - w_R, p_R^s - w_I]$ equations (7) and (8) hold, so $u_I = \frac{w_R + p_I^v - p_R^s}{w_R - w_I} b_I - \frac{p_I^v}{2}$, and, over this range, u_I is maximized either at $p_I^v = p_R^s - w_R > 0$ which implies $\pi = 1$ (because if $\pi < 1$ then tendering votes to I yields less than tendering shares to R, so cannot happen in equilibrium) or at $p_I^v = p_R^s - w_I$ which implies $u_I = b_I - \frac{p_I^v}{2} < 0$.

(B) If $b_R > 2b_I$ and $w_R < w_I$ ($w_R > w_I$ is covered by the preceding case), then R can start with $p_R^v > 2b_I$ and win profitably. To see this observe first that it would not be profitable for I to respond with $p_I^v \geq p_R^v$. Suppose that I responds with p_I^s. Clearly $p_I^s < w_R + p_R^v$ result in $\pi = 1$ and $p_I^s > w_I + p_R^v$ leads to losses for I. Otherwise (11) holds, $w_R \leq p_I^s - p_R^v \leq w_I$, π is given by (12) and $u_I = \frac{p_I^s - w_R - p_R^v}{w_I - w_R} b_I - \frac{p_R^v}{2}$. Hence $p_R^v > 2b_I$ implies that $u_I < 0$, which means that I's best response is to let R win.

(C) Suppose $w_I + b_I < w_R + b_R$ and $b_I < b_R$. First we argue that if $w_I > w_R$ then it cannot be that $\pi = 0$. If R offers $p_R^s \in (w_I + b_I, w_R + b_R)$ then I has no profitable counter offer and R has profits. To see that I has no profitable counter offer first note that $p_I^s \geq p_R^s > w_I + b_I$ then all tender to I so this cannot lead to gains for I. Next, if $p_I^v < p_R^s - w_I$ then $\pi = 1$. If $p_I^v > p_R^s - w_I$ then by the Pareto undomination part of the robustness requirement all shareholders tender votes to I and $u_I = b_I - p_I^v < b_I + w_I - p_R^s < 0$. If $p_I^v = p_R^s - w_I$ and not everyone sells to I and I wins then I may have a profit. But this is ruled out by the tie-free part of the robustness requirement.

If $w_R > w_I$ then it cannot be that $\pi = 0$. If R offers $p_R^v \in (b_I, b_R)$ then I has no profitable counter offer and R has profits. To see that I has no profitable counter offer first note that $p_I^v > p_R^v$ can only lead to losses. If $p_I^s < w_R + p_R^v$ then (due to the Pareto undomination part of the robustness requirement) I loses. If $p_I^s > w_R + p_R^v$ then all shareholders sell to I and I has losses. Finally, if $p_I^s = w_R + p_R^v$ and not everyone sells to I and I wins then I may have a profit. But this is ruled out by tie-free part of the robustness requirement.

(A) and (B) together cover cases (i) and (ii) while (C) covers (iii). ■

Proposition 4 *If (i) $w_R + b_R < w_I + b_I$ and $b_R < 2b_I$ or (ii) $b_R < b_I$ and $w_R + b_R < w_I + 2b_I$ then it cannot be that R wins.*

Proof. First consider the case $w_R + b_R < w_I + b_I$ and $b_R < 2b_I$. If $\pi = 1$ then either $p_R^s \geq w_I + b_I$ or $p_R^v \geq b_I$. (Otherwise I has a profitable deviation.) But if $p_R^s \geq w_I + b_I$ then, with $\pi = 1$ all shareholders tender shares to R so R has a loss, since $w_R + b_R < w_I + b_I$. If $p_R^v \geq b_I$ then there are two possibilities. If $w_R > w_I$, in which case $b_I > b_R$ (since $w_R + b_R < w_I + b_I$), then $p_R^v > b_R$ and with $\pi = 1$ all tender votes to R and that implies again that R has a loss. If $w_R < w_I$ then I can set p_I^s just below $w_I + p_R^v$ and win profitably with (just above) half the shareholders selling to I which is profitable for I while R has a loss. This proves (i).

If (ii) holds (but not (i)) then $b_R < b_I$ and $w_I + b_I < w_R + b_R < w_I + 2b_I$ so $w_R > w_I$ and if $\pi = 1$ then either $p_R^v \geq b_I > b_R$ and all tender votes to R and R has losses, or $p_R^s > w_R + b_R$ and all tender shares to R and R has losses, or $p_R^s \leq w_R + b_R$. But then if I offers $p_I^v = p_R^s - w_I - \varepsilon < 2b_I$ the only equilibrium in the tendering subgame is mixed with $\pi \approx 0$ (since if all tender votes to I it is better to tender shares to R [$p_R^s > p_I^v + w_I$] and if all tender shares to R it is better to tender ones vote to I [as $p_I^v + w_R > p_R^s$] so the equilibrium in the tendering subgame must be mixed with $p_R^w = p_I^v + \pi w_R + (1-\pi)\, w_I$ so that $p_I^v \approx p_R^s - w_I \Rightarrow \pi \approx 0$) and this is profitable to I. ■

Proof. (of Theorem 2): To see how the result follows from Lemma 3 and Propositions 3 and 4 we partition the parameter space as follows. Cases 2 and 4 below are those that correspond to cases 1 and 2 in the statement of the theorem.

1. $w_R + b_R < w_I + b_I$ and $b_R < 2b_I$ where I wins.

2. $w_R + b_R < w_I + b_I$ and $b_R > 2b_I$ where R wins.

3. $w_R + b_R > w_I + b_I$ and $b_I < b_R$ where R wins.

4. $w_I + 2b_I > w_R + b_R > w_I + b_I$ and $b_I > b_R$ where I wins.

5. $w_R + b_R > w_I + 2b_I$ $(> w_I + b_I)$ and $b_I > b_R$ where R wins.

By Lemma 3 and the existence result, in all equilibria either R wins or I wins with probability 1. Then Proposition 3 part (i) implies 5, part (ii) implies 2 (and part of 3), and part (iii) implies part 3. Proposition 4 part (i) implies 1, part (ii) implies 4 (and part of 1). ■

10.2 Proofs for subsection 5.1

Theorem 3 In all equilibria the contender with the higher value of $w_j + 2b_j$ wins.

Proof. First we observe that, without loss of generality we can restrict attention to I's offers $(p_I^s, \overline{m}_I^s)$ with $\overline{m}_I^s = 1/2$. To see this observe that, for given levels of m_I^s and π, any offer $(p_I^s, \overline{m}_I^s)$ is equivalent for shareholders to $(\hat{p}_I^s, 1/2)$, where $\hat{p}_I^s$ satisfies

$$(\hat{p}_I^s - [\pi w_R + (1-\pi)\, w_I]) \min\left\{\frac{1}{2m_I^s}, 1\right\} = (p_I^s - [\pi w_R + (1-\pi)\, w_I]) \min\left\{\frac{\overline{m}_I^s}{m_I^s}, 1\right\} \quad (14)$$

Therefore, there exists an equilibrium in the tendering subgame following $(\hat{p}_I^s, 1/2)$ with the same m_I^s and π.

Let u_I denote I's profit with $(p_I^s, \overline{m}_I^s)$

$$u_I = (1-\pi) b_I + \min(\overline{m}_I^s, m_I^s)[\pi w_R + (1-\pi)\, w_I - p_I^s]$$

and let $\hat{u}_I$ denote I's profit with $(\hat{p}_I^s, 1/2)$ and the same π

$$\hat{u}_I = (1-\pi)b_I + \min(1/2, m_I^s)[\pi w_R + (1-\pi)\, w_I - \hat{p}_I^s]$$

From $\min(x, m_I^s) = m_I^s \min[(x/m_I^s), 1]$ and (14), it follows that $u_I = \hat{u}_I$. Therefore, there exists an equilibrium in the tendering subgame following $(\hat{p}_I^s, 1/2)$ at which I gets the same profit as in the equilibrium of the tendering subgame following $(p_I^s, \overline{m}_I^s)$.

If $\pi \in (0,1)$ arises at equilibrium, it must be that $m_R^s = 1/2$, $\overline{m}_R^s \geq 1/2$ and $m_I^s \leq 1/2$. It cannot be that $m_I^s = 1/2$ and that I is over-subscribed because then fewer than 1/2 tender to R and I wins. If $\overline{m}_R^s = 1/2$ and R is oversubscribed then R wins (by our specification above – see remark 1). But then it cannot be that shareholders are selling shares to both I and R since this is ruled out by the tie-free part of the robustness requirement. Thus shareholders must be indifferent between selling to R and not tendering at all, implying that $p_R^s = \pi w_R + (1-\pi) w_I$. Let u_j denote the profit of $j = I, R$ in the putative equilibrium with $\pi \in (0,1)$.

$$u_I = (1-\pi)b_I = (1-\pi)b_I \tag{15}$$

$$u_R = \frac{1}{2}[-p_R^s + \pi w_R + (1-\pi)w_I] + \pi b_R = \pi b_R \tag{16}$$

Consider the following two configurations of parameters.

1. Suppose $w_I + 2b_I > w_R + 2b_R$ and that $\pi > 0$. It may not be that $\pi = 1$, since R's profitability implies $p_R^s \leq w_R + 2b_R$, but then I can win profitably with $(p_I^s, \overline{m}_I^s) = (\max\{p_R^s, w_R\}, 1/2)$. Thus, $\pi < 1$ in any equilibrium.

   Suppose then that $\pi \in (0,1)$, so that (15) and (16) hold. Consider a deviation by I to the offer $(p_I^s, \overline{m}_I^s) = (p_R^s + \varepsilon, 1/2)$, where ε is positive and small, say $\varepsilon < \pi 2b_R$. Contender I will end up buying from a mass $\theta \leq 0.5$ of the shareholders and win (since either $m_I^s > 1/2$, and I wins, or $m_I^s \leq 1/2$ which implies that nobody would tender to R since tendering to I is more profitable). Let $\hat{u}_I$ denote I's profit following this deviation:

$$\begin{aligned}
\hat{u}_I &= \theta(-p_R^s - \varepsilon + w_I) + b_I \\
&\geq \theta[-p_R^s - \varepsilon + \pi(w_R + 2b_R) + (1-\pi)(w_I + 2b_I)] + (1-2\theta)b_I \\
&= \theta[-\varepsilon + \pi 2b_R + (1-\pi)2b_I] + (1-2\theta)b_I > (1-\pi)b_I = u_I
\end{aligned}$$

where the first inequality follows from the assumption $w_I + 2b_I > w_R + 2b_R$. Thus, I can deviate profitably from the putative equilibrium with $\pi \in (0,1)$. Together with the previous observation that $\pi < 1$, we have that there is no equilibrium with $\pi > 0$. Combining this with the result on existence we conclude that with these parameters $\pi = 0$.

2. Suppose $w_I + 2b_I < w_R + 2b_R$ and that $\pi < 1$. It may not be the case that $\pi = 0$, since $p_R^s > \max\{w_I + 2b_I, w_R\}$ and $\overline{m}_R^s = 1/2$ would guarantee a profitable win for R, which I can defeat only at a loss. Therefore, $\pi \in (0,1)$ and again $p_I^s \leq p_R^s = \pi w_R + (1-\pi) w_I$ and (15) and (16) hold. Since it is an equilibrium, I cannot profitably outbid R with $(p_I^s, \overline{m}_I^s) = (p_R^s + \varepsilon, 1/2)$. That is,

$$u_I \geq b_I + (w_I - p_R^s)/2$$

Since $p_R^s = \pi w_R + (1-\pi) w_I$, this implies

$$u_I \geq b_I + (w_I - [\pi w_R + (1-\pi) w_I])/2 = (1-\pi) b_I + \pi (w_I + 2b_I - w_R)/2$$

If $w_I + 2b_I > w_R$, it follows that $u_I > (1-\pi) b_I$ in contradiction to (15). If $w_I + 2b_I \leq w_R$, then $\pi \in (0,1)$ may not arise in equilibrium, since $p_R^{s\prime} > w_R$ would guarantee R a win with profit $b_R + w_R - p_R^{s\prime}$. But, for $\hat{p}_R^s$ sufficiently close to w_R, $b_R + w_R - \hat{p}_R^s > \pi b_R \geq u_R$ in contradiction to equilibrium. Therefore, $\pi \in (0,1)$ cannot arise in equilibrium. Thus there is no equilibrium with $\pi < 1$. Combining this with the result on existence we conclude that with these parameters $\pi = 1$.

■

10.3 Proofs for subsection 5.2

Theorem 4 The identity of the winner is the same as in Theorem 3 except for parameter configurations satisfying $w_I + 2b_I > w_R + 2b_R$ and $b_R > b_I$. For these configurations I is the efficient contestant and would be the winner in the absence of vote trading, but R wins when vote trading is allowed.

Proof. The proof follows from the subsequent characterization of equilibrium outcomes and existence. By Lemma 4 and existence $\pi \in \{0,1\}$. Propositions 5 and 6 preclude either $\pi = 0$ or $\pi = 1$ for all possible configurations of the parameters. ■

Before proving that in equilibrium $\pi \notin (0,1)$ it is useful to establish that it suffices to restrict attention only to a subset of the possible offers, specifically to I making an offer $(p_I^s, 1/2; 0, 0)$ or $(0, 0; p_I^v, 1/2)$ and to R making an offer $(p_R^s, \overline{m}_R^s; p_R^v, \overline{m}_R^v)$ with $\overline{m}_R^s \geq 1/2$ and $\overline{m}_R^v \geq 1/2$. The next two claims formalize this result.

Claim 6 *For any $\pi \in (0,1)$ that arises in some tendering subgame following some f_R, f_I there exists an equilibrium in the subgame following f_R in which I's offer is $(p_I^s, 1/2; 0, 0)$ or $(0, 0; p_I^v, 1/2)$ and the subsequent tendering subgame (following f_R and I's offer of $(p_I^s, 1/2; 0, 0)$ or $(0, 0; p_I^v, 1/2)$) has the same π Moreover if the original equilibrium in the tendering subgame is not Pareto dominated by any strict equilibrium in the tendering subgame then neither is the equilibrium following f_R and I's offer of $(p_I^s, 1/2; 0, 0)$ or $(0, 0; p_I^v, 1/2)$ that has the same π.*

Proof. Suppose that I's offer in the original equilibrium is $(p_I^s, \overline{m}_I^s; p_I^v, \overline{m}_I^v)$. If shareholders tender to I only shares (i.e., $m_I^s > 0$ and $m_I^v = 0$), this offer is equivalent to $(p_I^s, \overline{m}_I^s; 0, 0)$. For the shareholders this is obviously equivalent to $(p_I^{s\prime}, 1/2; 0, 0)$, where $p_I^{s\prime}$ satisfies

$$(p_I^{s\prime} - [\pi w_R + (1-\pi)\, w_I]) \min[(1/(2m_I^s)), 1] = (p_I^s - [\pi w_R + (1-\pi)\, w_I]) \min[(\overline{m}_I^s / m_I^s), 1].$$

I's profit with $(p_I^s, \overline{m}_I^s; 0, 0)$ is

$$(1-\pi) b_I + \min(\overline{m}_I^s, m_I^s)[\pi w_R + (1-\pi)\, w_I - p_I^s]$$

Since π remains the same with $(p_I^{s\prime}, 1/2; 0, 0)$, I's profit with $(p_I^{s\prime}, 1/2; 0, 0)$ is

$$(1-\pi) b_I + \min(1/2, m_I^s)[\pi w_R + (1-\pi)\, w_I - p_I^{s\prime}]$$

Since $\min(\overline{m}^s, m_I^s) = m_I^s \min[(\overline{m}^s / m_I^s), 1]$, it follows that $(p_I^s, \overline{m}_I^s; 0, 0)$ and $(p_I^{s\prime}, 1/2; 0, 0)$ are equivalent for I as well.

An analogous argument would establish that, if shareholders tender to I only votes (i.e., $m_I^s = 0$ and $m_I^v > 0$), there is an equivalent offer $(0, 0; p_I^{v\prime}, 1/2)$.

Suppose therefore that shareholders tender to I both votes and shares (i.e., $m_I^s > 0$ and $m_I^v > 0$). This implies that they are indifferent between these two options. That is, $\pi w_R + (1-\pi)\, w_I + \min\{\overline{m}_I^v / m_I^v, 1\}\, p_I^v = \min\{\overline{m}_I^s / m_I^s, 1\}\, p_I^s + (1 - \min\{\overline{m}_I^s / m_I^s, 1\}) \times [\pi w_R + (1-\pi)\, w_I]$.

Clearly, the offer $(p_I^s, \overline{m}_I^{s\prime}; 0, 0)$ such that $\overline{m}_I^{s\prime} = \min\{\overline{m}_I^s(m_I^s + m_I^v)/m_I^s, 1\}$ is equivalent for the shareholders if $m_I^s + m_I^v$ tender to it. To see that it is also equivalent for I, observe that I's profit with $(p_I^s, \overline{m}_I^{s\prime}; 0, 0)$ equals

$$\begin{aligned}
& (1-\pi)b_I + \min\{\overline{m}_I^{s\prime}, m_I^s + m_I^v\}[\pi w_R + (1-\pi) w_I - p_I^s] \\
= \; & (1-\pi)b_I + (m_I^s + m_I^v)\min\{\overline{m}_I^{s\prime}/(m_I^s + m_I^v), 1\}[\pi w_R + (1-\pi) w_I - p_I^s] \\
= \; & (1-\pi)b_I + (m_I^s + m_I^v)\min[\min\{\overline{m}_I^s/m_I^s, 1/(m_I^s + m_I^v)\}, 1][\pi w_R + (1-\pi) w_I - p_I^s] \\
= \; & (1-\pi)b_I + m_I^s \min\{\overline{m}_I^s/m_I^s, 1\}[\pi w_R + (1-\pi) w_I - p_I^s] \\
& + m_I^v \min\{\overline{m}_I^s/m_I^s, 1\}[\pi w_R + (1-\pi) w_I - p_I^s] \\
= \; & (1-\pi)b_I + m_I^s \min\{\overline{m}_I^s/m_I^s, 1\}[\pi w_R + (1-\pi) w_I - p_I^s] - m_I^v \min\{\overline{m}_I^v/m_I^v, 1\} p_I^v \\
= \; & (1-\pi)b_I + \min\{\overline{m}_I^s, m_I^s\}[\pi w_R + (1-\pi) w_I - p_I^s] - \min\{\overline{m}_I^v, m_I^v\} p_I^v
\end{aligned}$$

which equals I's profit with $(p_I^s, \overline{m}_I^s; p_I^v, \overline{m}_I^v)$.

The second equality follows from the definition of $\overline{m}_I^{s\prime}$, the third from $1/(m_I^s + m_I^v) \geq 1$, and the fourth from the shareholders' indifference.

Finally, it follows from the previous argument that $(p_I^s, \overline{m}_I^{s\prime}; 0, 0)$ is equivalent to $(p_I^{s\prime}, 1/2; 0, 0)$.

It is straightforward to verify that the equilibrium in a tendering subgame that is constructed in this proof is not Pareto dominated by any strict equilibrium in the subgame if the original equilibrium in the tendering subgame was not Pareto dominated. ■

Claim 7 *For any $\pi \in (0,1)$ that arises in some tendering subgame following f_R, f_I there exists an equilibrium in the tendering subgame following an offer by R, $(p_R^s, \overline{m}_R^s; p_R^v, \overline{m}_R^v)$, that satisfies $\overline{m}_R^s \geq 1/2$ or $\overline{m}_R^v \geq 1/2$ and which has the same $\pi \in (0,1)$. Moreover if the original equilibrium in the tendering subgame is not Pareto dominated by a strict equilibrium in the tendering subgame then neither is the equilibrium of the tendering subgame that has the same π and follows the aforementioned restricted offers.*

Proof. Consider the case $\overline{m}_R^s < 1/2$ and $\overline{m}_R^v < 1/2$. It has to be that $\overline{m}_R^s + \overline{m}_R^v \geq 1/2$, since otherwise $\pi = 0$. Since $\pi \in (0,1)$, at least one of R's offers is not oversubscribed, for otherwise R would win. If offer p_R^s is not oversubscribed, then the offer $(p_R^s, 1/2; p_R^v, \overline{m}_R^v)$ when coupled with the same response by I would leave the existing shareholders' tendering decisions optimal, hence would yield the same π and the same payoffs for R and I. And, if I has a better response against $(p_R^s, 1/2; p_R^v, \overline{m}_R^v)$ than its original response, then this response would be also better against the original offer by R. An analogous argument can be made if it is p_R^v that is not oversubscribed, in which case the offer $(p_R^s, \overline{m}_R^s; p_R^v, 1/2)$ would achieve the same result against I's response.

We also need to argue why this construction does not violate the Pareto undomination part of the robustness requirement. If $w_R > p_R^s$ then the only equilibrium in the tendering subgame has 1/2 selling to R; this is unchanged. If $w_R \leq p_R^s$ then if all sell to R they get $w_R/2 + p_R^s/2 \leq p_R^s$ which they get in the constructed equilibrium of the tendering subgame. ■

Lemma 4 *There is no equilibrium in which both R and I have a strictly positive probability of winning, i.e., there is no equilibrium with $\pi \in (0,1)$.*

Proof. Suppose $\pi \in (0,1)$. This implies that R ends up acquiring exactly half votes (with or without shares) and that shareholders are indifferent between tendering to R and the alternative of tendering to I or keeping their shares. That is, $\min\{m_R^s, \overline{m}_R^s\} + \min\{m_R^v, \overline{m}_R^v\} = 1/2$. By the preceding claim at least one of R's offers is not restricted to quantity below 1/2. That offer is not oversubscribed, since if it were R would win. Thus, there must be indifference between that offer and the same alternative as there was in the second sentence of this paragraph.

Given these observations, the proof mimics that of Lemma 3 essentially verbatim. ■

Proposition 5 *If $w_R + 2b_R > w_I + 2b_I$, or $b_R > b_I$, then I cannot win.*

Proof. If $w_R + 2b_R > w_I + 2b_I$, or $b_R > b_I$, it may not be that $\pi = 0$, since in the former case R can start with $(p_R^s, 1/2; 0, 0)$ such that $p_R^s \in (w_I + 2b_I, w_R + 2b_R)$ and in the latter case with $(0, 0; p_R^v, 1/2)$ such that $p_R^v > 2b_I$ and win profitably in both cases. ■

Proposition 6 *If $w_R + 2b_R < w_I + 2b_I$ and $b_R < b_I$, then R cannot win.*

Proof. If R wins with probability 1 then either $p_R^s \geq w_I + 2b_I$ and $\overline{m}_R^s \geq 1/2$, or $p_R^v \geq 2b_I$ and $\overline{m}_R^v \geq 1/2$. In both cases R has losses, so there is no such equilibrium. ■

10.4 Proofs for subsection 6.1

The following lemma narrows down the set of scenarios that have to be considered.

Lemma 5 *Given any robust equilibrium with outcome π there is a robust equilibrium with outcome π when we restrict attention to the case where I makes only uncontingent offers, and R does not make both types of offers, only one.*

Proof. We first argue that wlog attention can be restricted to the case where I makes only uncontingent offers. Consider then the case in which I makes a contingent offer p_I^{sc}. In a mixed equilibrium of the tendering subgame the shareholders would be indifferent either between tendering to R and to I or between tendering to R and just holding on to the shares. In the former case the payoff to a shareholder from tendering to I would be $(1-\pi)p_I^{sc}+\pi w_R$ and the payoff to I would be $(1-\pi)b_I+\theta(1-\pi)(w_I-p_I^{sc})$, where $\theta\in[0,1/2]$ is the fraction of shares tendered to I. It follows that, if I offers instead the non-contingent price $p_I^{s\prime}=(1-\pi)p_I^{sc}+\pi w_R$, the above outcome will continue to be an equilibrium of the tendering subgame. That is, the probability of R's win will continue to be π, a fraction θ will tender to I and those tendering to I and those who do not will receive the same payoff. I's payoff will be $(1-\pi)b_I+\theta[\pi w_R+(1-\pi)\,w_I-p_I^{s\prime}]=(1-\pi)b_I+\theta(1-\pi)(w_I-p_I^{sc})$ just as before. Thus, in a mixed equilibrium, without loss of generality, we may assume that I is confined to making only non-contingent offers. So it is enough to examine contingent offers only by R.

The Pareto dominance part of the refinement might rule out an equilibrium with $\pi\in(0,1)$ under p_I^{sc} but not for $p_I^s=(1-\pi)\,p_I^{sc}+\pi w_R$. However, this does not affect the argument just given, since whenever the Pareto dominance part of the refinement would rule out an equilibrium with $\pi\in(0,1)$ for $p_I^s=(1-\pi)\,p_I^{sc}+\pi w_R$ it would also rule it out for p_I^{sc}. The constructed equilibrium will satisfy the tie-free requirement as well since ties were not used in the construction, so if one happens to be created nearby actions will be tie-free and have approximately the same π.

When $\pi\in\{0,1\}$ it is obvious that I can be confined to non-contingent offers wlog—if without being confined I loses then I continues to lose with a restricted strategy space; if without being confined I wins with probability 1 then the contingent offer is equivalent to an uncontingent offer. Clearly in these cases the new strategies constitute a (robust) equilibrium.

Now we argue that wlog attention can be restricted to the case where R does not make both contingent and non-contingent offers, just one of the two. If $\pi\in(0,1)$, shareholders must be indifferent between R's contingent offer and I's uncontingent offer (since the tie-free part of the robustness implies that they do not tender to the uncontingent offers of both) and hence they must prefer these to R's uncontingent offer (i.e., $\pi p_R^{sc} + (1-\pi)\, w_I = p_I^s \geq p_R^s$ and no shares are tendered to R at p_R^s). Hence R's contingent offer is what shareholders tender to so the uncontingent offer by R is then irrelevant. If R loses with probability 1 then restricting R's strategy space is clearly wlog. If R wins with probability 1 then replacing any contingent offer with an uncontingent one will not change shareholder or I's behavior. That the constructed equilibrium is robust is obvious. ∎

Theorem 5 If $w_k + b_k > w_j + b_j$ then in all equilibria k wins.

Proof. The method of the proof is again to rule out mixed equilibria in which both contenders win with positive probability. Recall that in such a putative mixed equilibrium the shareholders are just indifferent about tendering to R and exactly half tender to R. We know from the analysis in section 4.1 that there is no such equilibrium when both contenders make non-contingent offers. We have now to extend this conclusion to the cases in which at least one contender makes a conditional offer and the shareholders are indifferent between such an offer and an alternative.

Consider therefore the case in which R makes a contingent offer p_R^{sc} and I responds with a non-contingent offer p_I^s. In a mixed equilibrium of the tendering subgame, it may not be that $p_R^{sc} < w_I$, since then this outcome would fail robustness due to Pareto domination by the strict equilibrium in the subgame in which shareholders hold on to their shares. Therefore, $p_R^{sc} \geq w_I$. In a mixed equilibrium of the subgame the shareholders would be indifferent either between tendering to R and tendering to I or between tendering to R and just holding on to the shares. The latter case is ruled out since it implies $\pi p_R^{sc} + (1-\pi)\, w_I = \pi w_R + (1-\pi)\, w_I$, hence $p_R^{sc} = w_R$, which is not consistent with $\pi \in (0,1)$ and the tie-free condition of robustness.

In the former case $\pi p_R^{sc} + (1-\pi)\, w_I = p_I^s$ so that $\pi = \frac{p_I^s - w_I}{p_R^{sc} - w_I}$ and $u_I = (1-\pi)\, b_I + (\pi w_R + (1-\pi)\, w_I - p_I^s)\, \theta = b_I + \frac{p_I^s - w_I}{p_R^{sc} - w_I}\left((w_R - p_R^{sc})\, \theta - b_I\right)$, where $\theta \leq 1/2$ is the fraction selling to I. Now, if $\left(\frac{w_R - p_R^{sc}}{2} - b_I\right) > 0$, then u_I is increasing in p_I^s so I will set $p_I^s = p_R^{sc}$ resulting in $\pi = 1$. If $\left(\frac{w_R - p_R^{sc}}{2} - b_I\right) < 0$, then u_I is decreasing in p_I^s so I will set $p_I^s = w_I$ resulting in $\pi = 0$. Thus, in either case $\pi \in \{0, 1\}$.

The rest of the proof is as in the case of non-contingent offers. ∎

10.5 Proofs for subsection 6.2

Theorem 6 The efficient contender wins in equilibrium except in the following regions of the parameter space.

1. If $w_I + b_I > w_R + b_R$ and $b_R > 2b_I$, then R wins.
2. If $w_I + b_I < w_R + b_R < w_I + 2b_I$ and $b_I > b_R$, then I wins.

Proof. The proof is like that of Theorem 2. It follows from the subsequent characterization of equilibrium outcomes and existence. By Lemma 7 and existence $\pi \in \{0, 1\}$. Propositions 7 and 8 preclude either $\pi = 0$ or $\pi = 1$ for all possible configurations of the parameters . For example, part 1 follows from Proposition 7 part (ii). ■

Before proving that in all equilibria $\pi \notin (0, 1)$, we present a result analogous to Lemma 5 showing that for our purposes we can restrict attention to a subset of the strategy space.

Lemma 6 *The equilibrium value of π is unchanged if we restrict attention to the case where I makes only uncontingent offers, and R does not make both contingent and uncontingent offers for shares, nor both contingent and uncontingent offer for votes, i.e., $p_R^v \times p_R^{cv} = 0$ and $p_R^s \times p_R^{cs} = 0$.*

Proof. The proof follows exactly the same lines as that of Lemma 5. The only change is that if there is an equilibrium in which I offers $p_I^{vc} > 0$ we must show that there is an alternative equilibrium in which $p_I^{vc} = 0$. This follows since instead of offering p_I^{vc} I could offer $p_I^v = (1 - \pi)\, p_I^{vc}$. When offering p_I^{vc} the payoffs to shareholders tendering votes to I conditionally would be $(1 - \pi)p_I^{vc} + (1 - \pi)\, w_I + \pi w_R$ and the payoff to I would be $(1 - \pi)b_I + \theta(1 - \pi)(-p_I^{vc})$, where $\theta \in [0, 1/2]$ is the fraction of shares tendered to I. With $p_I^v = (1 - \pi)p_I^{vc}$, the same outcome will continue to be an equilibrium of the tendering subgame. This is because given the same π those tendering to I and those who do not will receive the same payoff and I's payoff will be $(1-\pi)b_I + \theta\,(-p_I^{s\prime}) = (1-\pi)b_I + \theta(1-\pi)(-p_I^{sc})$ just as before. ■

Lemma 7 *With conditional (but unrestricted) offers for shares and votes there is no equilibrium in which I and R both have a strictly positive probability of winning, i.e., there is no equilibrium with $\pi \in (0, 1)$.*

Proof. For $\pi \in (0,1)$ it must be that shareholders tender shares to one contender and votes to the other.

The tendering of uncontingent shares both to I and to R is precluded by the tie-free part of the robustness. Tendering of uncontingent shares to I and contingent shares to R is precluded by the following argument. If this were the case we would have $\pi p_R^{sc} + (1-\pi) w_I = p_I^s$. It may not be that $p_R^{sc} = w_I = p_I^s$, since then the tie-free part of the robustness would rule out tendering to both. So, it has to be either $w_I < p_I^s < p_R^{sc}$ or $w_I > p_I^s > p_R^{sc}$. But both of these cases are ruled out by the Pareto domination part of the robustness requirement. In the first case, the putative equilibrium outcome in the tendering subgame is Pareto dominated by all tendering to R which is a strict equilibrium in the tendering subgame (note that $p_R^{sc} \geq w_R$ or else there will be no tendering to R in the first place). Consider then the second case and a $(f_R^\varepsilon, f_I^\varepsilon)$ as required by the robustness condition. If $p_I^{v\varepsilon} > p_R^{v\varepsilon}$ then the equilibrium where all shareholders tender votes to I is a strict equilibrium that Pareto dominates the the original outcome π. If $p_R^{v\varepsilon} > p_I^{v\varepsilon}$ then it must be that $p_R^{sc} > w_R$ as otherwise it is not an equilibrium for shareholders to sell shares to R as selling votes to R yields more ($\pi p_R^{sc} + (1-\pi) w_I < \pi w_R + (1-\pi) w_I + p_R^v$). But then I's profits are $\frac{1}{2}(\pi w_R + (1-\pi) w_I) - \frac{1}{2} p_I^s + (1-\pi) b_I$ which equals $\frac{1}{2}\pi (w_R - p_R^{sc}) + (1-\pi) b_I$ (by substituting $\pi p_R^{sc} + (1-\pi) w_I = p_I^s$) which is decreasing in π in which case the optimal p_I^s is equal to w_I whereupon $\pi = 0$.

The same type of arguments rule out the sale of votes to both I and to R.

Finally, there cannot be an equilibrium with $\pi \in (0,1)$ in which some shareholders tender to R and some do not tender at all. The impossibility of some not tendering and some tendering shares for uncontingent prices was demonstrated in Lemma 1. That they cannot be indifferent between selling votes at uncontingent or contingent prices and not tendering is obvious. The possibility of some tendering to a contingent offer by R and some not tendering when $p_R^{cs} = w_R$ is ruled our by the tie-free part of the robustness requirement.

Given that wlog contenders do not make both a conditional and unconditional offer for shares nor make both conditional and unconditional offers for votes, the preceding discussion implies that if $\pi \in (0,1)$ then one of the following must hold.

1. $\pi p_R^{sc} + (1-\pi) w_I = \pi w_R + (1-\pi) w_I + p_I^v$

2. $\pi p_R^{sc} + (1-\pi) w_I = \pi w_R + (1-\pi) w_I + (1-\pi) p_I^{vc}$

3. $p_R^s = \pi w_R + (1-\pi) w_I + p_I^v$

4. $p_R^s = \pi w_R + (1-\pi)\, w_I + (1-\pi)\, p_I^{vc}$

5. $\pi p_R^{vc} + \pi w_R + (1-\pi)\, w_I = p_I^s$

6. $\pi p_R^{vc} + \pi w_R + (1-\pi)\, w_I = (1-\pi)\, p_I^{sc} + \pi w_R$

7. $p_R^v + \pi w_R + (1-\pi)\, w_I = p_I^s$

8. $p_R^v + \pi w_R + (1-\pi)\, w_I = (1-\pi)\, p_I^{sc} + \pi w_R$

We consider these cases next. For cases 1–4, as in Lemma 3, if $w_I > w_R$ then in the tendering subgame the strict equilibrium in which all tender to I (which one an easily verify *is* an equilibrium of the tendering subgame when the relevant equality condition in 1, 2, 3 or 4, is satisfied) Pareto dominates for shareholders any equilibrium of the tendering subgame with $\pi \in (0,1)$. So the robustness requirement implies that $\pi \notin (0,1)$. Hence in 1–4 we only consider the case $w_I < w_R$.

i. If $p_R^{sc} > w_R + p_I^v$ then all sell to R by the Pareto undomination part of the robustness requirement. If $p_R^{sc} < w_R + p_I^v$ then the only equilibrium of the tendering subgame is for all to sell to I. Hence if $p_R^{sc} \neq w_R + p_I^v$ we have $\pi \notin (0,1)$. In the case $p_R^{sc} = w_R + p_I^v$ the tie-free part of the robustness implies $\pi \notin (0,1)$.

ii. Given any equilibrium of this type with some $\pi \in (0,1)$ we can construct an equilibrium of type 1 with $p_I^v = (1-\pi)\, p_I^{vc}$ since then payoffs to shareholders and to I and R are the same. Since no equilibrium of type 1 with $\pi \in (0,1)$ exists, the same conclusion applies to equilibria of type 2. (There is also a simple direct argument: $p_R^{sc} > w_R$ since otherwise no one sells to R. Since $w_R > w_I$ all selling to R – which is an equilibrium of the tendering subgame – is better than any payoff with $\pi \in (0,1)$ so by the Pareto undomination part of the robustness requirement $\pi \notin (0,1)$. The case of $\pi \in (0,1)$ arising due to $p_R^{sc} = w_R$ is ruled out by the tie-free part of the robustness requirement.

iii. This situation is identical to the case studied in Lemma 3 of $\pi \in (0,1)$ without conditional offers, and therefore is not feasible for $\pi \in (0,1)$.

iv. The same argument as in case 2, but applied to case 3, implies that there is no equilibrium with $\pi \in (0,1)$ in case 4.

We turn now to cases 5–8. As discussed in Lemma 3 $w_R > w_I$ implies that the Pareto undomination part of the robustness requirement selects the equilibrium in the tendering subgame where all sell to R. So we consider $w_I > w_R$.

v. Assume there is an interior solution for π (otherwise we are done with this step).

If $p_I^s < w_R + p_R^{vc}$ then all selling to R is the only equilibrium outcome of the tendering subgame that survives the Pareto undomination part of the robustness requirement.

If $p_I^s > w_R + p_R^{vc}$ then, since we are assuming there is an interior solution for π we also must have $w_I > p_I^s$ (by the equality in condition 5). Then $\pi = \frac{w_I - p_I^s}{w_I - p_R^{vc} - w_R}$ and $u_I = (1-\pi) b_I + (\pi w_R + (1-\pi) w_I - p_I^s) \theta = (1-\pi) b_I - \pi p_R^{vc} \theta$, where $\theta \leq 1/2$ is the fraction of conditional votes purchased by I. This is decreasing in π hence increasing in p_I^s. So the optimal solution for I is at $\pi = 0$.

If (*) $p_I^s = w_R + p_R^{vc}$ then by the tie-free part of the robustness requirement $\pi \notin (0,1)$.

vi. The argument in the proof of Lemma 6 implies that we can assume wlog that I does not make conditional price offers. Hence the proof in part 5 applies to this case. (There is also a simple direct argument: $p_I^{sc} \geq w_I$ since otherwise no one sells to I. Since $w_I > w_R$ all selling to I – which is an equilibrium in the tendering subgame – is better than any payoff with $\pi \in (0,1)$ so, if $p_I^{sc} > w_I$, by the the Pareto undomination part of the robustness requirement refinement $\pi \notin (0,1)$. The case $p_I^{sc} = w_I$ and $\pi \in (0,1)$ is ruled out by the tie-free part of the robustness requirement.)

vii. This is the same as in the unconditional analysis of Lemma 3.

viii. The argument in the proof of Lemma 6 again implies that we can assume wlog that I does not make conditional price offers. Hence the proof in part 7 applies to this case. (There is also a simple direct argument: If $p_I^{sc} > w_I + p_R^v$ then all sell to I by the Pareto undomination part of the robustness requirement. If $p_I^{sc} < w_I + p_R^v$ then the only equilibrium in the tendering subgame is for all to sell to R. Hence if $p_I^{sc} \neq w_I + p_R^v$ we have $\pi \notin (0,1)$. The case of $\pi \in (0,1)$ due to $p_I^{sc} = w_I + p_R^v$ is ruled out by the tie-free part of the robustness requirement.)

■

Proposition 7 *If (i)* $w_R+b_R > w_I+2b_I$*, or (ii)* $b_R > 2b_I$*, or (iii) both* $w_I+b_I < w_R+b_R < w_I+2b_I$ *and either* $w_I > w_R$ *or* $b_I < b_R < 2b_I$ *then* I *cannot win in any equilibrium.*

Proof. The proof of parts (i) and (ii) exactly mimics parts A and B in the proof of Proposition 3, except that in addition to considering I responding with p_I^v or p_I^s we also allow for responses of p_I^{vc} and p_I^{sc}. That is, $\pi = 0$ cannot arise in equilibrium since R can open with $p_R^s \in (\max\{w_I+2b_I, w_R\}, w_R+b_R)$ if condition (i) of the proposition holds, or with $p_R^v > 2b_I$ if condition (ii) of the proposition holds.

That against the former an offer of p_I^{sc} that wins with positive probability is not profitable holds for the same reason that an offer of p_I^s that wins with positive probability is not profitable. That an offer of p_I^{vc} that wins with positive probability is not profitable holds since when $p_I^{vc}+w_I \geq p_R^s$ if I wins then I has losses because $p_I^{vc} \geq p_R^s - w_I > 2b_I$, while if $p_I^{vc}+w_I < p_R^s$ all sell to R.

Against $p_R^v > 2b_I$ again it is clearly unprofitable for I to win with an offer of p_I^{vc} just as with an offer of p_I^v. An offer of $p_I^{sc} \geq w_I + p_R^v$ and I winning results in I having losses, while $p_I^{sc} < w_I + p_R^v$ results in all selling to R.

Similarly, the proof for part (iii) mimics part C in the proof Proposition 3. To be comprehensive we repeat it here and note that the same arguments work when I also can respond with p_I^{vc} and p_I^{sc}. If $w_I > w_R$ it cannot be that $\pi = 0$. If R offers $p_R^s \in (w_I+b_I, w_R+b_R)$ then I has no profitable counter offer and R has profits. To see that I has no profitable counter offer first note that $p_I^s > p_R^s$ can only lead to losses, and the same holds for p_I^{sc}. (If $p_I^s = p_R^s$ and I wins profitably then some, but not all, shareholders sell to I, but this is ruled out by the tie-free part of the robustness requirement.) If $p_I^v < p_R^s - w_I$ then $\pi = 1$. If $p_I^v > p_R^s - w_I$ then all shareholders tender to I and $u_I = b_I - p_I^v < b_I + w_I - p_R^s < 0$. (If $p_I^v = p_R^s - w_I$ and I wins profitably then some, but not all, shareholders sell to I but this is ruled out by the tie-free part of the robustness requirement.) The same holds for p_I^{vc}.

If $w_R > w_I$ then it cannot be that $\pi = 0$. If R offers $p_R^v \in (b_I, b_R)$ then I has no profitable counter offer and R has profits. To see that I has no profitable counter offer first note that $p_I^v \geq p_R^v$ and I winning can only lead to losses **for** I, and the same for p_I^{vc}. If $p_I^s < w_R + p_R^v$ then (due to the Pareto undomination part of the robustness requirement) I loses. If $p_I^s > w_R + p_R^v$ then all shareholders sell to I and I has losses, and the same holds for p_I^{sc}. (If $p_I^s = w_R + p_R^v$ and I wins profitably then some, but not all, shareholders sell to I but this is ruled out by the tie-free part of the robustness requirement.) ■

Proposition 8 *If $w_R + b_R < w_I + b_I$ and $b_R < 2b_I$ or $b_R < b_I$ and $w_R + b_R < w_I + 2b_I$ then R cannot win.*

Proof. The proof mimics that of Proposition 4. The only difference is that R may open with $p_R^{vc} \geq b_I$. In this case setting $p_I^s = w_I + p_R^{vc}$ (analogous to the behavior after $p_R^v \geq b_I$) is not profitable for I as due to the contingent nature of R's offer, all will tender to I. However we have that $p_R^{vc} < b_R$ (since otherwise if R wins with probability 1 then R has losses), and then if I sets p_I^s just above $w_R + p_R^{vc}$ everyone sells to I and this is profitable to I. ■

Remark 6 *The parameter regions considered in Propositions 7 and 8 include all possible configurations, but they are not a partition of the parameter space; for example (i) and (ii) of Proposition 7 overlap.*

11 References

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