THE PINHAS SAPIR CENTER FOR DEVELOPMENT
TEL AVIV UNIVERSITY

Asymmetric Contests with Interdependent Valuations"

Ron Siegel[1]

Discussion Paper No. **7-**11

October 2011

---

[1] Ron Siegel – Department of Economics, Northwestern University, Evanston, IL 60208.
Email: r-siegel@northwestern.edu
I thank Tel Aviv University for its hospitality and the Pinhas Sapir Center for Development for financial support. I thank Alvaro Parra for outstanding research assistantship. I thank Renato Gomes for numerous detailed and thoughtful comments. I thank Jeremy Bulow, Johannes Horner, Benny Moldovanu, Eddie Dekel, Wolfgang Leininger, Ilya Segal, Bruno Strulovici, Jeroen Swinkels, Asher Wolinsky and seminar participants at CSIO-IDEI, Dortmund, Northwestern, the Second Brazilian Game Theory Workshop, Tel Aviv University, the Tel Aviv International Workshop on Game Theory and Toronto for helpful comments and suggestions.

Abstract:

I show that a unique equilibrium exists in an asymmetric two-player all-pay auction with a discrete signal structure that satisfies a monotonicity condition in each player's signal. Independent signals and asymmetric interdependent valuations are a special case. The proof is constructive, and the construction is simple to implement as a computer program. For special cases, which include some private value settings, common value settings, and symmetric players, I derive additional properties and comparative statics. I also characterize the set of equilibria when a reserve price is introduced.

affect both contestants' valuation for the prize, and contestants are asymmetric in that their private information may be drawn from an asymmetric distribution and impact their valuations differently. For example, consider a research and development race in which the firm with the higher-quality product enjoys a dominant market position. Each firm may be informed about different attributes of the market, which together determine the value of winning. This value may differ between the firms, because the profit associated with a dominant market position may depend on firm-specific characteristics such as production costs and marketing expertise. Similar asymmetries in information and valuations for the prize arise in rent-seeking scenarios, such as lobbying, and in other competitions with sunk investments, such as competitions for promotions.

Section 2 models the contest as an asymmetric all-pay auction with interdependent valuations. Each player privately observes a signal drawn from a finite ordered set, and these sets may differ between the players. After observing his signal, each player decides how much to bid, both players pay their bids, and the player with the higher bid wins the prize. The value of the prize is a player-specific function of both players' signals. In addition to a full support assumption, the only restriction I impose is a joint monotonicity condition on players' valuations and the distribution of players' signals (Condition M). The condition requires that for each player and every signal of the other player, the product of the player's valuation and the conditional probability of the other player's signal increase in the player's signal.[1] This condition does not directly restrict how a player's valuation is affected by the other player's signal. Players' signals may or may not be affiliated (see Section 2.2 for an example). In the special case of independent signals, the condition simplifies to the requirement that a player's valuation increase in his own signal for every signal of the other player. This is automatically satisfied when players have private values, and is not required when signals are not independent. The model includes complete information, private values, common values, and one informed and one uninformed player as special cases.

Section 3 contains the main result of the paper, which is a constructive characterization

[1] Throughout the paper, by "increase," "decrease," "positive," and "negative" I mean "strictly increase," "strictly decrease," "strictly positive," and "strictly negative."

of the unique equilibrium. I begin by constructing the unique candidate for a *monotonic* equilibrium, in which higher types choose bids from higher intervals. This ordering of intervals means that, by proceeding from the top, the candidate equilibrium can be constructed in a finite number of steps. In each step, one type of player 1 "competes" against one type of player 2.[2] In the resulting interval of competition, the players behave as in a complete-information all-pay auction with valuations that correspond to the competing types. Once one player has exhausted his probability mass, any remaining probability mass of the other player is expended as an atom at 0. This simple procedure is easy to implement as a computer program whose input is players' valuation functions and the distribution of players' signals (three two-dimensional matrices) and whose output is players' strategies (two vectors).[3]

For this candidate equilibrium to be an equilibrium, it suffices that the monotonicity condition hold with "weakly increase" instead of "increase" (Condition WM). This is always the case, for example, when one player has no private information. Even when this weak version of the monotonicity condition fails, the candidate equilibrium may still be an equilibrium, because not all of the requirements entailed by the condition necessarily bind (see Section 3.2 for an example). If the monotonicity condition (Condition M) holds, then the outcome of the procedure is the *unique* equilibrium, because the monotonicity condition implies that any equilibrium is monotonic.

Section 4 applies the construction procedure to examine a few special cases. First, a closed-form solution is provided when players have private values, one player is known to be stronger than the other, and only one player has private information. When the privately-informed player is the strong one, his low types enjoy a higher payoff increase relative to the corresponding complete-information contest than do his high types. A first-order stochastic dominance (FOSD) shift in his type distribution increases his expected payoff and causes the weak player to bid less aggressively, in a FOSD sense. When the privately-informed player is the weak one, his high types enjoy a weakly higher payoff

[2] A player's type is the signal he observes.

[3] A Matlab implementation of this procedure is available on my website, http://faculty.wcas.northwestern.edu/~rsi665/.

increase relative to the corresponding complete-information contest than do his low types. A FOSD shift in his type distribution may increase or decrease his payoff, and decreases the payoff of the strong player. Second, a partial characterization is provided when the value of the prize is common to both players. This characterization shows that players' equilibrium bid distributions are identical from an ex–ante perspective. Players' payoffs may differ, however, because each player can condition his bid on his private information, which may differ between the players. A closed form solution is provided when, in addition to common values, only one player has private information. If this player becomes more informed, then both players become less aggressive in a FOSD sense, which decreases overall expenditures and increases the informed player's payoff. Third, a symmetric closed-form solution is provided when players are "quasi-symmetric," in that whenever they observe the same signals the conditional probabilities of their signals are the same and their valuations for winning are the same.

Section 5 extends the model by adding a reserve price, which corresponds to a minimum investment necessary to win the contest. A player who bids below the reserve price loses, regardless of what the other player bids. Under the monotonicity condition, the structure of any equilibrium is closely related to that of the unique equilibrium without a reserve price. In particular, there exists a bid such that in any equilibrium, players' bidding behavior above the reserve price coincides with their bidding behavior above this bid in the contest without a reserve price. There may be multiple equilibria, which differ in the probabilities that players bid 0 and the reserve price. I characterize the set of equilibria, which are payoff equivalent, and show that players' payoffs weakly decrease in the reserve price. Any two equilibria differ in the behavior of at most one type for each player, so when the probability of each type is small the difference between any two equilibria is small. This is consistent with Lizzeri and Persico's (2000) result that with a continuum of types and a sufficiently high reserve price there exists a unique monotonic equilibrium. Appendix C contains examples of contests with a reserve price and their equilibria.

A key assumption of the model is that each player's set of possible signals is finite. This assumption shows that certain insights and techniques used in the analysis of complete-information all-pay auctions apply when there is incomplete information, which provides a novel connection between complete and incomplete information all-pay auctions. Complete-information all-pay auctions are a special case of the model, in contrast to their usual treatment as a limiting case in models with a continuum of signals and atomless distributions. The model also allows for one informed and one uninformed player, with private, interdependent, or common values. The finiteness assumption helps overcome many of the technical difficulties that plague existence and uniqueness proofs in models with atomless signal distributions, and facilitates equilibrium characterization with a reserve price.[4] Moreover, for any fixed number of possible signals, it is straightforward to derive from the construction procedure a closed form solution for the equilibrium. This solution depends on the possible equilibrium orderings of players' bidding intervals, which are determined by players' valuation functions and the distribution of players' signals. Appendix B enumerates the possible equilibrium orderings, and provides a complete characterization of the equilibrium when each player has one or two possible signals (excluding the case of complete information, which has been well studied by, for example, Hillman and Riley (1989)). This characterization generalizes the ones obtained by Konrad (2004, 2009) and (independently from this paper) by Szech (2011), who examined two-player all-pay auctions with independent private values and one or two types.

In contrast to this paper, most of the literature on all-pay auctions with incomplete information assumes a continuum of signals and atomless distributions.[5] The three most relevant papers in this literature, all of which make this assumption, are Morgan and Krishna (1997), Lizzeri and Persico (2000), and Amann and Leininger (1998). Morgan and Krishna (1997) studied the multiplayer war of attrition and the all-pay auction in a setting with affiliated signals, a symmetric signal distribution, and symmetric valuations that increase in both players' signals, and obtained for the two-player all-pay auction a closed-form solution for the unique monotonic equilibrium. Lizzeri and Persico (2000) studied a general model of asymmetric two-player bidding games in a setting with affiliated signals and a reserve price that is high enough to exclude a positive measure of types for each

[4]This characterization applies, of course, to complete-information all-pay auctions with a reserve price, which are analyzed in Appendix C, and, to the best of my knowledge, have not been studied previously.

[5]Exceptions are the aforementioned papers by Konrad (2004,2009) and Szech (2011).

player from bidding, and obtained for the two-player all-pay auction an existence result for the unique monotonic equilibrium. Both papers require for these results a continuous version of the monotonicity condition discussed above, but neither obtains an unqualified equilibrium uniqueness result. In contrast, this paper derives an unqualified, constructive equilibrium uniqueness result, which does not require affiliation, symmetry, monotonicity of a player's valuation in the other player's signal, or the existence of a reserve price. Amann and Leininger (1996) studied an asymmetric two-player all-pay auction with independent private values, and obtained a constructive characterization of the unique candidate for a differentiable, monotonic equilibrium.[6] More recently, Parreiras and Rubinchik's (2010) characterized some equilibrium properties of an asymmetric all-pay auction with a continuum of signals and atomless distributions, independent private values, and more than two players.

2 Model

There are two players and one prize. Each player $i = 1, 2$ observes a private signal s_i, which I refer to as the player's type. Player i's set of possible signals, S_i, is a finite set of cardinality $n_i > 0$. The elements in S_i are ordered from high to low according to a strict ranking $\succ_i$, so $s_i^1 \succ_i s_i^2 \succ_i \cdots \succ_i s_i^{n_i}$. I denote by $\succ$ the pair of rankings $(\succ_1, \succ_2)$. The signals in $S_1 \times S_2$ are distributed according to a probability distribution with probability mass function f that has full support, so $f(s_1, s_2) > 0$ for every (s_1, s_2) in $S_1 \times S_2$. Abusing notation, let $f(s_i) = \sum_{s_{-i} \in S_{-i}} f(s_1, s_2)$, where $-i$ refers to player $3 - i$, and denote by $f(s_i | s_{-i}) = f(s_1, s_2) / f(s_{-i})$ the conditional probability of player i's signal s_i given player $(-i)$'s signal s_{-i}. The full support assumption guarantees that all conditional probabilities are well defined and positive.

After observing their signals, the players compete in an all-pay auction. They simultaneously choose how much money to bid, forfeit their bids, and the player with the higher

[6] They did not prove that the candidate equilibrium is indeed an equilibrium, or that it is unique within the class of monotonic equilibria. These lacunae can most likely be filled by the tools developed in Lizzeri and Persico (2000).

bid wins the prize (in case of a tie, any procedure can be used to allocate the prize between the players). Player i's valuation for the prize is $V_i : S_i \times S_{-i} \to \mathbb{R}_{++}$.[7] Thus, if the players bid b_1 and b_2, and their signals are s_1 and s_2, then player i's payoff is

$$P_i(b_i, b_{-i}) V_i(s_i, s_{-i}) - b_i,$$

where

$$P_i(b_i, b_{-i}) = \begin{cases} 1 & \text{if } b_i > b_{-i}, \\ 0 & \text{if } b_i < b_{-i}, \\ \text{any value in } [0,1] & \text{if } b_i = b_{-i}, \end{cases},$$

such that $P_1(b_1, b_2) + P_2(b_2, b_1) = 1$. The sets of possible signals S_i, the distribution f, and the valuation functions V_i are commonly known.

The following monotonicity condition on players' valuations and conditional signal distributions will play an important role in the equilibrium analysis.

M For $i = 1, 2$, $f(s_{-i}|s_i) V_i(s_i, s_{-i})$ increases in s_i for every s_{-i}.

Condition M implies that player i's expected valuation for the prize,

$$E_{s_{-i}} V_i(s_i, s_{-i}) = \sum_{s_{-i} \in S_{-i}} f(s_{-i}|s_i) V_i(s_i, s_{-i}),$$

increase in s_i. But the condition is more restrictive than this: it requires that every component in the sum increase in s_i. Note that Condition M places no direct restrictions on how V_i changes with s_{-i}.[8] While $V_i(s_i, s_{-i})$ may increase in s_i for every s_{-i} (but does not have to), the same is not true of $f(s_{-i}|s_i)$, because f is a probability distribution. Condition M specifies the degree to which $f(s_{-i}|s_i)$ may decrease in s_i when $V_i(s_i, s_{-i})$

[7] As in Szech (2011), player's signals could affect their constant marginal costs of bidding instead of or in addition to their effect on players' valuations.

[8] A continuous version of Condition M appeared in Morgan and Krishna (1997) and Lizzeri and Persico (2000). Both papers also required players' signals to be affiliated; In addition, Krishna and Morgan (1997) required each player's valuation to increase in both signals, and Lizzeri and Persico (2000) required the introduction of a sufficiently high reserve price (see the discussion in Section 5). These requirements are not made here.

increases in s_i. For example, if increasing a player's signal increases the player's valuation by a multiplicative factor of at least $\alpha > 1$, then any signal distribution for which the same signal increase does not decrease the conditional probability of the other player's signal by a multiplicative factor of α or more satisfies Condition M. This is illustrated by the example in Section 2.2.

The following condition is a specialization of Condition M to the case of independent signals.

IM Players have independent signals, and for $i = 1, 2$, $V_i(s_i, s_{-i})$ increases in s_i for every s_{-i}.

Condition IM implies Condition M, and holds, for example, when players have independent private values, or independent signals and common values that increase in both players' signals. Condition M generalizes Condition IM by relaxing the independence and monotonicity assumptions, and requiring instead only a form of joint monotonicity. A further relaxation of the monotonicity requirement leads to the following condition.

WM For $i = 1, 2$, $f(s_{-i}|s_i) V_i(s_i, s_{-i})$ weakly increases in s_i for every s_{-i}.

Conditions WM and M play different roles in the equilibrium analysis. Condition WM guarantees that the output of the construction procedure described in Section 3 is an equilibrium, while Condition M guarantees that the equilibrium is unique. Although the conditions are similar, the following example describe a setting in which Condition WM is naturally satisfied but Condition M is not.

2.1 Example 1

Suppose that player 1's valuation is known to be 1, and player 2's valuation is 1 or 2 with equal probability. The twist is that player 2's valuation is known only by player 1. That is, player 1's signal equals player 2's valuation, $f(1) = f(2) = 1/2$, and player 2 has only one signal, s_2. Condition WM is satisfied (and Condition M fails) for player 1, regardless of whether $1 \succ_1 2$ or $2 \succ_1 1$, because $f(s_2|1) = f(s_2|2) = 1$ and $V_1(1, s_2) = V_1(2, s_2) = 1$. Condition WM is trivially satisfied for player 2, because he has only one signal.

2.2 Example 2

Consider a private value setting in which player 1's valuation is either 1 or $2d$, and player 2's valuation is either 3 or $4d$, for some fixed $d \geq 1$. Each player's signal equals his valuation, $2d \succ_1 1$, $4d \succ_2 3$, and

$$f(2d, 4d) = f(1, 3) = \frac{1}{2} - \varepsilon,\ f(2d, 3) = f(1, 4d) = \varepsilon,$$

for some ε in $(0, 1/2)$. Player's valuations are perfectly correlated for $\varepsilon = 0$, perfectly negatively correlated for $\varepsilon = 1/2$, statistically independent for $\varepsilon = 1/4$, and affiliated for $\varepsilon \leq 1/4$.[9] We have that

$$f(1) = f(3) = f(2d) = f(4d) = \frac{1}{2},$$

$$f(2d|4d) = f(4d|2d) = f(1|3) = f(3|1) = 1 - 2\varepsilon, \tag{1}$$

and

$$f(2d|3) = f(3|2d) = f(1|4d) = f(4d|1) = 2\varepsilon. \tag{2}$$

For $i = 1$, Condition M is

$$f(3|2d)\, 2d > f(3|1)\, 1 \iff 4\varepsilon d > 1 - 2\varepsilon \iff \varepsilon > \frac{1}{4d + 2}$$

and

$$f(4d|2d)\, 2d > f(4d|1)\, 1 \iff (1 - 2\varepsilon)\, 2d > 2\varepsilon \iff \varepsilon < \frac{d}{2d + 1}.$$

For $i = 2$, Condition M is

$$f(1|4d)\, 4d > f(1|3)\, 3 \iff 8d\varepsilon > 3 - 6\varepsilon \iff \varepsilon > \frac{3}{8d + 6}$$

[9] When $\varepsilon \leq 1/4$, players' valuations can be viewed as conditionally independent as follows. Consider a signal whose possible realizations are L, M, and H with probabilities $1/2 - 2\varepsilon$, 4ε, and $1/2 - 2\varepsilon$. When the realization is L, player 1's valuation is 1 and player 2's valuation is 3; when the realization is M, players' valuations are distributed independently and uniformly; when the realization is H, player 1's valuation is $2d$ and player 2's valuation is $4d$.

and

$$f\left(2d|4d\right)4d > f\left(2d|3\right)3 \iff \left(1-2\varepsilon\right)4d > 6\varepsilon \iff \varepsilon < \frac{2d}{4d+3}.$$

Because $1/\left(4d+2\right) < 3/\left(8d+6\right)$ and $2d/\left(4d+3\right) < d/\left(2d+1\right)$, Condition M is satisfied for ε in $\left(3/\left(8d+6\right), 2d/\left(4d+3\right)\right)$, and Condition WM is satisfied for ε in $\left[3/\left(8d+6\right), 2d/\left(4d+3\right)\right]$. That is, when players' valuations are not too positively or negatively correlated. Note that $1/4$ is in this range. This corresponds to independent private values, for which Condition IM is satisfied. As d increases, the range of values of ε for which Condition M is satisfied approaches $(0, 1/2)$.

3 Equilibrium

Denote a mixed strategy of player i by $G_i : S_i \times \mathbb{R} \to [0,1]$, where $G_i\left(s_i, x\right)$ is the probability that player i bids at most x when his type is s_i (so $G_i\left(s_i, \cdot\right)$ is a cumulative distribution function (CDF) for every signal s_i). Abusing notation, I will sometimes suppress the first argument and use G_i to denote player i's ex-ante mixed strategy, unconditional of his type, so $G_i\left(x\right)$ is the probability that player i bids at most x ($G_i\left(\cdot\right) = \sum_{s_i \in S_i} f\left(s_i\right) G_i\left(s_i, \cdot\right)$). Denote by $BR_i\left(s_i\right)$ player i's set of best responses when his type is s_i and the other player plays G_{-i}. Condition M implies that higher types have higher best response sets, regardless of the other player's strategy. This is the content of the following lemma, whose proof, like all other omitted proofs, is in Appendix A.

Lemma 1 *If Condition M holds and $s_i' \succ_i s_i$, then for any x in $BR_i\left(s_i\right)$ and y in $BR_i\left(s_i'\right)$ we have $x \leq y$.*

An equilibrium is a pair $\mathbf{G} = \left(G_1, G_2\right)$, such that $G_i\left(s_i, \cdot\right)$ assigns measure 1 to $BR_i\left(s_i\right)$, for every signal s_i. When higher types have higher best response sets, we have a *monotonic equilibrium*.

Definition 1 *An equilibrium $\mathbf{G}$ is* monotonic *if for $i = 1, 2$ and any $s_i' \succ_i s_i$, x in $BR_i\left(s_i\right)$ and y in $BR_i\left(s_i'\right)$ implies $x \leq y$.*

Because in equilibrium best responses are chosen with probability 1, in any monotonic equilibrium higher types choose higher bids. An immediate implication of Lemma 1 is the following.

Corollary 1 *If Condition M holds, then any equilibrium is monotonic.*

I begin by enumerating properties of any equilibrium, monotonic or not.[10] I say that a player has an *atom* at x if the player bids x with positive probability.

Lemma 2 *In any equilibrium, (i) there is no bid at which both players have an atom, (ii) there is no positive bid at which either player has an atom, (iii) if a positive bid is not a best response for player i for any signal, then no weakly higher bid is a best response for either player for any signal, and (iv) both players have best responses at* 0 *or arbitrarily close to* 0.

The remainder of the section constructs the unique candidate for a monotonic equilibrium, and shows that Condition WM suffices for this candidate to be an equilibrium. To this end, suppose that a monotonic equilibrium exists, and denote it by $\mathbf{G}$. The following lemma characterizes players' best response sets.

Lemma 3 *For $i = 1, 2$ and any s_i, $BR_i(s_i)$ is an interval. For any two consecutive signals $s_i' \succ_i s_i$, the upper bound of $BR_i(s_i)$ is equal to the lower bound of $BR_i(s_i')$. Moreover,*

$$\sup \cup_{s_1 \in S_1} BR_1(s_1) = \sup \cup_{s_2 \in S_2} BR_2(s_2) \;\; \textit{and} \;\; \inf \cup_{s_1 \in S_1} BR_1(s_1) = \inf \cup_{s_2 \in S_2} BR_2(s_2) = 0. \tag{3}$$

Figure 1 depicts an equilibrium structure consistent with Lemma 3, where T denotes the common upper bound of players' best response sets.[11]

[10] Similar equilibrium properties arise in many complete-information models of competition, such as those of Bulow and Levin (2006) and Siegel (2009, 2010).

[11] Note that because $T > 0$ (at most one player has an atom at 0) and players' strategies are continuous above 0 (part (ii) of Lemma 2), T is a best response for both players' highest types.

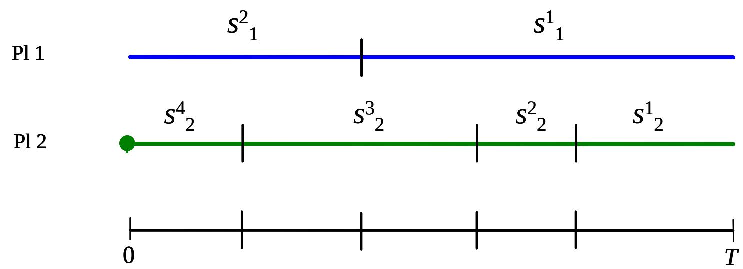

Figure 1: A possible structure of players' best response sets in a monotonic equilibrium, when player 1 has two signals, player 2 has four signals, and player 2 has an atom at 0

This structure shows that the equilibrium can be found by starting from the top and using an iterative procedure (without knowing the value of T in advance). To see this, consider the coarsest partition of $[0,T]$ into intervals that includes both partitions of $[0,T]$ into players' best response sets (henceforth: the joint partition). In Figure 1, the joint partition is depicted on the bottom line. Consider two bids $x < y$ in the top interval of this partition. Both x and y are best responses for player 1 when his type is s_1^1 (recall that s_i^k is player i's k^{th} signal when his signals are ordered from high to low), and therefore lead to the same expected payoff. That is,

$$\begin{aligned} & \sum_{s_2 \prec_2 s_2^1} \left(f\left(s_2|s_1^1\right) V_1\left(s_1^1, s_2\right)\right) + f\left(s_2^1|s_1^1\right) V_1\left(s_1^1, s_2^1\right) G_2\left(s_2^1, y\right) - y \\ = & \sum_{s_2 \prec_2 s_2^1} \left(f\left(s_2|s_1^1\right) V_1\left(s_1^1, s_2\right)\right) + f_2\left(s_2^1|s_1^1\right) V_1\left(s_1^1, s_2^1\right) G_2\left(s_2^1, x\right) - x, \end{aligned} \tag{4}$$

which can be rewritten as

$$\frac{G_2\left(s_2^1, y\right) - G_2\left(s_2^1, x\right)}{y - x} = \frac{1}{f\left(s_2^1|s_1^1\right) V_1\left(s_1^1, s_2^1\right)}.$$

Taking $y - x$ to 0 shows that in the top interval $G_2\left(s_2^1, \cdot\right)$ is differentiable with constant density

$$g_2\left(s_2^1, \cdot\right) = \frac{1}{f\left(s_2^1|s_1^1\right) V_1\left(s_1^1, s_2^1\right)}.$$

Similarly, in the top interval $G_1\left(s_1^1, \cdot\right)$ is differentiable with constant density

$$g_1\left(s_1^1, \cdot\right) = \frac{1}{f\left(s_1^1|s_2^1\right) V_2\left(s_2^1, s_1^1\right)}.$$

(Note that these densities generalize the ones that arise in the unique equilibrium of the complete-information all-pay auction (Hillman and Riley (1989)), which are, respectively, $1/V_1$ and $1/V_2$, where V_i is player i's commonly-known valuation for the prize.)

Having identified the densities of players' strategies in the top interval of the joint partition, we can find the length of this interval. For this, note that, as in Figure 1, in the top interval of the joint partition (at least) one of the two players exhausts the probability mass associated with his highest type (i.e., his highest type does not choose bids below this interval). Therefore, the length of the top interval is

$$\min\left\{f\left(s_2^1|s_1^1\right)V_1\left(s_1^1,s_2^1\right),f\left(s_1^1|s_2^1\right)V_2\left(s_2^1,s_1^1\right)\right\}, \tag{5}$$

with the player whose density determines the length of the interval exhausting the probability mass associated with his highest type. Players' densities in the next interval are calculated in a similar fashion, with the player(s) who has exhausted the probability mass associated with his highest type "moving" to his second highest type. This process is iterated, calculating the length of each interval and players' densities in each interval. Suppose we are in the k^{th} (from the top) interval of the joint partition, after player 1 has exhausted the probability mass associated with his k_1 highest types and player 2 has exhausted the probability mass associated with his k_2 highest types, so type $s_1^{k_1+1}$ of player 1 "competes" against type $s_2^{k_2+1}$ of player 2. The equivalent of (4) for player 1 is then

$$\begin{aligned}&\sum_{s_2\prec_2 s_2^{k_2+1}}\left(f\left(s_2|s_1^{k_1+1}\right)V_1\left(s_1^{k_1+1},s_2\right)\right)+f\left(s_2^{k_2+1}|s_1^{k_1+1}\right)V_1\left(s_1^{k_1+1},s_2^{k_2+1}\right)G_2\left(s_2^{k_2+1},y\right)-y\\=&\sum_{s_2\prec_2 s_2^{k_2+1}}\left(f\left(s_2|s_1^{k_1+1}\right)V_1\left(s_1^{k_1+1},s_2\right)\right)+f\left(s_2^{k_2+1}|s_1^{k_1+1}\right)V_1\left(s_1^{k_1+1},s_2^{k_2+1}\right)G_2\left(s_2^{k_2+1},x\right)-x,\end{aligned}$$

and similarly for player 2, which leads to constant densities

$$g_2\left(s_2^{k_2+1},\cdot\right)=\frac{1}{f\left(s_2^{k_2+1}|s_1^{k_1+1}\right)V_1\left(s_1^{k_1+1},s_2^{k_2+1}\right)}\text{ and }g_1\left(s_1^{k_1+1},\cdot\right)=\frac{1}{f\left(s_1^{k_1+1}|s_2^{k_2+1}\right)V_2\left(s_2^{k_2+1},s_1^{k_1+1}\right)}. \tag{6}$$

When computing the length of the k^{th} interval, the probability mass associated with types $s_1^{k_1+1}$ and $s_2^{k_2+1}$ expended on higher intervals must be taken into account (at most one of these signals will have probability mass expended on higher intervals, by definition of the joint partition). The length of the k^{th} interval is the minimal length required for some player

i to exhaust the (remaining) probability mass associated with his type $s_i^{k_i+1}$ when players' densities are given by (6). Appendix B enumerates the possible equilibrium orderings of players' types induced by this procedure, and provides a complete characterization of the equilibrium when each player has one or two possible types (excluding the case of complete information).

When one of the players has exhausted the probability mass associated with his lowest type, the remaining mass of the other player must be an atom, and this atom must be at 0 (part (ii) of Lemma 2). This atom may include the mass associated with several types. If both players exhaust their probability mass simultaneously, then the point of exhaustion is also 0 (part (iv) of Lemma 2). By going from 0 upwards, the equilibrium can be constructed from players' densities on each interval. The value of T is the sum of the lengths of the intervals that make up the joint partition. The reason that the construction can proceed from the top without knowing the value of T in advance is that the equilibrium densities at any given bid, given by (6), depend only on the types for which the bid is a best response, and not on the bid itself.[12] This is due to the all-pay feature, and is not true, for example, in a first-price auction.[13]

That the construction produces a unique outcome while relying on properties of any monotonic equilibrium proves the following result.

Lemma 4 *The procedure above constructs the unique candidate for a monotonic equilibrium. Each player's strategy is continuous above* 0 *and piecewise uniform. At most one player has an atom, at* 0*.*

The construction guarantees that no local deviations exist in the interior of any interval of the joint partition. Condition WM rules out other deviations, as the following result shows.

Proposition 1 *If Condition WM holds, then the outcome of the procedure above is an equilibrium.*

[12] A similar property enables the equilibrium construction in Bulow and Levin (2006).

[13] Asymmetric first-price auctions have been studied extensively in the economics literature. For a useful recent literature review see Mares and Swinkels (2009).

Proposition 1 does not rule out the existence of other equilibria. Indeed, the procedure may output a different equilibrium for each ranking $\succ$ of players' signals,[14] and there may be additional equilibria as well. This is demonstrated by the example in Section 3.1. Condition M rules out such multiplicity, because it guarantees that any equilibrium is monotonic (Corollary 1). We therefore have the following result.

Proposition 2 *If Condition M holds, then the procedure above constructs the unique equilibrium, which is monotonic.*

Proof. By Corollary 1, any equilibrium is monotonic (relative to the given ranking). The outcome of the procedure is the only candidate for this equilibrium, by Lemma 4. This is an equilibrium, by Proposition 1.[15] ■

Corollary 2 *If Condition IM holds, then the procedure above constructs the unique equilibrium. In particular, the procedure above constructs the unique equilibrium when players have independent private values.*

Proof. The first part of the corollary is immediate from Proposition 2, because Condition IM implies Condition M. For the second part, note that with independent private values a player's signal does not affect the conditional distribution of the other player's signal or the other player's valuation. Therefore, it is without loss of generality to assume that V_i increases in s_i, so Condition IM holds. ■

If S_{-i} is a singleton, then Condition WM simplifies to requiring that V_i weakly increase in s_i. This clearly holds for some ranking of player i's signals. And by perturbing V_i slightly,

[14] These equilibria need not be monotonic, because the best response sets of different types may overlap, but they do have the property that higher types choose bids from higher intervals. See the example in Section 3.1.

[15] One can also take an indirect approach and apply an equilibrium existence result to prove that when Condition M holds the unique candidate for an equilibrium is indeed an equilibrium. For example, Simon and Zame's (2000) result implies that an equilibrium exists, because the finite number of types means that a player's pure strategy can be viewed as an element of a finite-dimensional Euclidean space, and the tie-breaking rule does not matter, because no ties arise in equilibrium (part (i) of Lemma 2). For a similar application of Simon and Zame's (2000) result see the proof of Siegel's (2009) Corollary 1.

if necessary, we obtain a contest in which V_i increases in s_i, so Condition M holds.[16] This proves the following result.

Corollary 3 *If one player has no private information, then there exists a ranking of the other player's signals such that the procedure above constructs an equilibrium. Moreover, slightly perturbing the informed player's valuation function makes the outcome of the procedure the unique equilibrium of the contest.*[17]

When both players have private information, Condition WM may not hold for any ranking of players' signals. As Section 2.2 shows, however, not all the inequalities required by Condition WM are necessarily "binding," in that the outcome of the construction may still be an equilibrium even though some inequalities fail. The binding inequalities are determined by the types that "compete" against each other, i.e., have overlapping best response sets. But because which types compete against each other is determined in equilibrium, the binding inequalities are not easy to identify in advance. Of course, as long as f has full support, for any ranking of players' signals the construction procedure produces the unique candidate for a monotonic equilibrium. It is then straightforward to check whether this candidate in indeed an equilibrium, by checking if any player has profitable deviations given the other player's strategy. It is then also readily verified which of the inequalities required by Condition WM bind, and which can be relaxed.

3.1 Example 1

Let us apply the construction procedure to the contest described in Section 2.1. For each of the two rankings of player 1's signals, the outcome is an equilibrium, because Condition WM holds (this is shown in Section 2.1).

For the ranking $2 \succ_1 1$, in the top interval player 1's type 2 competes against player 2 (who has only one type, s_2), so we have

$$g_1(2,\cdot) = \frac{1}{f(2|s_2)\,2} = \frac{1}{\left(\frac{1}{2}\right)2} = 1 \text{ and } g_2(s_2,\cdot) = \frac{1}{f(s_2|2)\,1} = \frac{1}{(1)\,1} = 1, \tag{7}$$

[16] As Section 2.1 shows, it may not be without loss of generality to assume that V_i increases in s_i.

[17] If neither player has private information, then we have a complete-information all-pay auction, whose unique equilibrium is constructed by the procedure.

and the length of the interval is 1. In this interval, player 1 exhausts the probability mass associated with his type 2, and player 2 exhausts his probability mass. Because player 2 has no probability mass left, the lower bound of the interval is 0, and player 1 expends the probability mass associated with his type 1 as an atom at 0. Therefore, $T = 1$.

For the ranking $1 \succ_1 2$, in the top interval player 1's type 1 competes against player 2, so we have

$$g_1(1,\cdot) = \frac{1}{f(1|s_2)\,1} = \frac{1}{\left(\frac{1}{2}\right)1} = 2 \text{ and } g_2(s_2,\cdot) = \frac{1}{f(s_2|1)\,1} = 1,$$

and the length of the interval is $1/2$. In this interval, player 1 exhausts the probability mass associated with his type 1, and player 2 expends $1/2$ of his probability mass. In the next interval, player 1's type 2 competes against player 2, so players' densities are given by (7). Given these densities, player 2 exhausts his remaining probability mass on an interval of length $1/2$, and player 1 exhausts the probability mass associated with his type 2 on an interval of length 1. Therefore, the length of the interval is $1/2$. In this interval, player 2 exhausts his remaining probability mass, and player 1 expends $1/2$ of the probability mass associated with his type 2. Because player 2 has no probability mass left, the lower bound of the interval is 0, and player 1 expends the remaining probability mass of $1/2$ associated with his type 2 as an atom at 0. The sum of the lengths of the two intervals is $T = 1/2 + 1/2 = 1$.

Another equilibrium is one in which player 1 ignores his signal, so the players compete as in the all-pay auction in which player 1's valuation is 1 and player 2's valuation is $3/2$. In this equilibrium, regardless of his type, player 1 mixes uniformly with density $2/3$ on $[0,1]$, and bids 0 with probability $1/3$. Player 2 mixes uniformly with density 1 on $[0,1]$.[18] This equilibrium disappears if the contest is perturbed slightly so that player 1's valuation depends on his signal. Such a perturbation leads to a unique equilibrium, because Condition M then holds for one of the rankings, $2 \succ_1 1$ or $1 \succ_1 2$. Which of the first two equilibria "survives" depends on whether player 1's valuation is higher when his signal is 2 (the first equilibrium) or 1 (the second equilibrium).

[18] In all three equilibria the best response set of each of player 1's types is $[0,1]$. In the first two equilibria, which are constructed by the procedure, the high type chooses higher bids than the low type.

3.2 Example 2

Let us apply the construction procedure to the contest described in Section 2.2. In the top interval, players' high types compete, so we have

$$g_1(2d,\cdot) = \frac{1}{f(2d|4d)\,4d} = \frac{1}{(1-2\varepsilon)\,4d} \text{ and } g_2(4d,\cdot) = \frac{1}{f(4d|2d)\,2d} = \frac{1}{(1-2\varepsilon)\,2d},$$

and the length of the interval is $(1-2\varepsilon)\,2d$. In this interval player 2 exhausts the probability mass associated with his high type, and player 1 expends $(1-2\varepsilon)\,2d/(1-2\varepsilon)\,4d = 1/2$ of the probability mass associated with his high type. In the next interval, the high type of player 1 and the low type of player 2 compete, so we have

$$g_1(2,\cdot) = \frac{1}{f(2d|3)\,3} = \frac{1}{6\varepsilon} \text{ and } g_2(3,\cdot) = \frac{1}{f(3|2d)\,2d} = \frac{1}{4\varepsilon d}.$$

Given these densities, player 1 exhausts the remaining probability mass associated with his high type on an interval of length $(1/2)/(1/6\varepsilon) = 3\varepsilon$, and player 2 exhausts the probability mass associated with his low type on an interval of length $4\varepsilon d$. Therefore, the length of the interval is 3ε. In this interval player 1 exhausts the remaining probability mass associated with his high type, and player 2 expends $3\varepsilon/4\varepsilon d = 3/4d$ of the probability mass associated with his low type. In the next interval, players' low types compete, so we have

$$g_1(1,\cdot) = \frac{1}{f(1|3)\,3} = \frac{1}{3-6\varepsilon} \text{ and } g_2(3,\cdot) = \frac{1}{f(3|1)\,1} = \frac{1}{1-2\varepsilon}.$$

Given these densities, player 1 exhausts the probability mass associated with his low type on an interval of length $3-6\varepsilon$, and player 2 exhausts the remaining probability mass associated with his low type on an interval of length $(1/4d)/(1/(1-2\varepsilon)) = (1-2\varepsilon)/4d$. Therefore, the length of the interval is $(1-2\varepsilon)/4d$. In this interval player 2 exhausts the remaining probability mass associated with his low type, and player 1 expends $(1-2\varepsilon)/(4d\,(3-6\varepsilon)) = 1/12d$ of the probability mass associated with his low type. Because player 2 has no more probability mass left, the lower bound of the interval is 0, and player 1 expends the remaining probability mass of $1-1/12d$ associated with his low type as an atom at 0. The sum of the lengths of the three intervals is

$$T = (1-2\varepsilon)\,2d + 3\varepsilon + \frac{1-2\varepsilon}{4d} = (1-2\varepsilon)\left(\frac{8d^2+1}{4d}\right) + 3\varepsilon.$$

The output of the construction procedure is depicted in Figure 2.

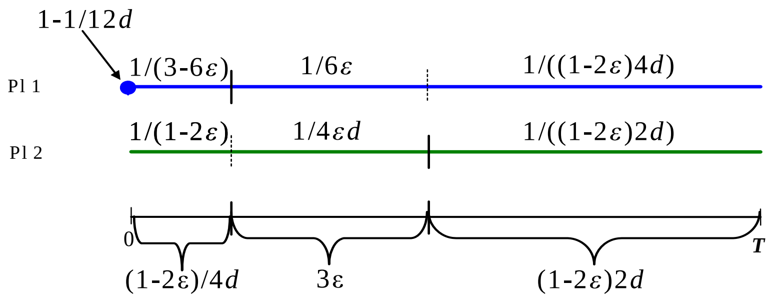

Figure 2: Players' densities and player 1's atom

By Proposition 2, Figure 2 depicts the unique equilibrium for ε in $(3/(8d+6), 2d/(4d+3))$, because for these values of ε Condition M holds (this is shown in Section 2.1). By Proposition 1, Figure 2 also depicts an equilibrium for ε in $\{3/(8d+6), 2d/(4d+3)\}$, because for these values of ε Condition WM holds. What about values of ε lower than $3/(8d+6)$? For ε in $[1/(4d+2), 3/(8d+6))$, Condition WM fails because $f(1|4d)4d < f(1|3)3$, but all the other inequalities required for Condition WM hold. Therefore, the only deviations to check are by player 2 when his valuation is $4d$ to bids in $(0, (1-2\varepsilon)/4d)$ (the bids made by player 1 when his valuation is 1 and by player 2 when his valuation is 3). Such deviations give player 2 a payoff no higher than $2\varepsilon(12d-1)/3$, the limiting payoff from bidding arbitrarily close to 0. But for $\varepsilon < 3/(8d+6)$ this payoff is lower than the payoff from bidding T. Therefore, for ε in $[1/(4d+2), 3/(8d+6))$ the output of the construction procedure is still an equilibrium, even though Condition WM fails. The reason for this is as follows. Although player 2 when his valuation is $4d$ obtains a higher payoff from bidding slightly above 0 than from bidding $(1-2\varepsilon)/4d$ (because $f(1|4d)4d < f(1|3)3$), he also obtains a higher payoff from bidding $3\varepsilon + (1-2\varepsilon)/4d$ than from bidding $(1-2\varepsilon)/4d$ (because $f(2d|4d)4d > f(2d|3)3$), and the increase in payoff from bidding slightly above 0 instead of $(1-2\varepsilon)/4d$ is smaller than the increase in payoff from bidding $3\varepsilon + (1-2\varepsilon)/4d$ instead of $(1-2\varepsilon)/4d$. Things are different for $\varepsilon < 1/(4d+2)$. In this case, $f(3|2d)2d < f(3|1)1$, and player 1 has profitable deviations. When his valuation is 1, he obtains more than 0 by bidding $3\varepsilon + (1-2\varepsilon)/4d$, but only 0 by bidding below $(1-2\varepsilon)/4d$; when his valuation is $2d$, he obtains 0 by bidding 0, but less than 0 by bidding above $(1-2\varepsilon)/4d$. For $\varepsilon > 2d/(4d+3)$, we have $f(2d|4d)\,4d < f(2d|3)\,3$, and player 2 has profitable deviations. When his valuation is $2d$, he obtains a higher payoff by bidding $(1-2\varepsilon)/4d$ than by bidding T. Similarly, bidding T is a profitable deviation for player 2 when his valuation is 3.

Therefore, the output of the construction procedure is an equilibrium for ε in $[1/(4d+2), 2d/(4d+3)]$ (and the equilibrium is unique for ε in $(3/(8d+6), 2d/(4d+3))$), and for ε in $[0, 1/(4d+2)) \cup (2d/(4d+3), 1/2]$ there is no monotonic equilibrium. For example, when $\varepsilon = 0$ there is a unique equilibrium, in which player 1's best response set is $[0,1]$ when his valuation is 1 and $[0,2d]$ when his valuation is $2d$, and player 2's best response set is $(0,1]$ when his valuation is 3 and $(0,2d]$ when his valuation is $4d$. This is because $\varepsilon = 0$ implies full correlation of players' signals, so players bid as in the complete-information all-pay auction that corresponds to their valuations.

4 Special Cases

Throughout this section, assume that Condition M holds, and denote by $\mathbf{G} = (G_1, G_2)$ the unique equilibrium (In Sections 4.1.1, 4.1.2, and 4.2.1, Condition M need not be assumed, because it is implied by Corollaries 2 and 3).

4.1 Independent Signals and Monotonic Valuation Functions

If players' signals are independent and a player's valuation function weakly increases in the other player's signal, then the other player's unconditional bid distribution is concave. That is, the other player's unconditional bid density is lower on higher intervals of the joint partition. To see why, note that for almost any x in $(0,T]$ we have

$$G_i(x) = \sum_{s_i \in S_i} f(s_i)\, G_i(s_i, x) \Rightarrow g_i(x) = f(s_i(x))\, g_i(s_i(x), x)$$

$$= \frac{f(s_i(x))}{f(s_i(x)\,|s_{-i}(x))\, V_{-i}(s_{-i}(x), s_i(x))} = \frac{1}{V_{-i}(s_{-i}(x), s_i(x))}, \tag{8}$$

where g_i is the density of G_i, $s_i(x)$ is the signal of player i for which x is a best response, and the last equality follows from independence of players' signals. Because $s_i(\cdot)$ is monotonic (Lemma 1), the monotonicity of g_i follows from that of V_{-i} (V_{-i} is monotonic in s_{-i} by Condition IM). It is clear from (8) that if a player's valuation function is not monotonic in the other player's signal, then the other player's strategy need not be concave, even if players' signals are independent. It is also true that if players' signals are not independent, then concavity may fail, even when players' valuations are monotonic.[19]

A natural comparative static to consider is a first-order stochastic dominance (FOSD) shift in a player's (marginal) signal distribution. Such a shift would seem to make the player "stronger," and therefore (weakly) increase his payoff and decrease the other player's payoff. This, however, is not always what happens. Even when Condition IM is satisfied and each player's valuation weakly increases in the other player's signal, the effects of a FOSD shift (or, similarly, an increase in a player's valuation function) depend qualitatively on the parametrization of the contest. The payoff of either player may increase or decrease, as may overall expenditures. Consider, for example, the case of complete information. It is easy to see that an increase in the valuation of the weak player (the one with the lower valuation) decreases the other player's payoff and increases expenditures, because is reduces the strong player's advantage and makes competition "more intense," while an increase in the valuation of the strong player increases his payoff and decreases expenditures. Figure 3 and the description that follows it show that a FOSD shift can decrease a player's expected payoff, because it makes competition "more intense." It is also not difficult to generate examples of contests in which a FOSD shift increases the other player's payoff.[20] Of course, unambiguous comparative statics, as well as closed form solutions, can be obtained for more restricted classes of contests. I now consider one such class, in which players have

[19]To see this, set $\varepsilon = 5/24$ in the equilibrium of Figure 2. Then player 1's ex-ante density is $2/7$ in the lowest interval of the joint partition, $2/5$ in the next interval, and $3/14$ in the top interval.

[20]One such example has independent signals, two signals for each player, player 1's private valuation equaling 2 with probability p and 1 with probability $1-p$, player 2's signal equalling 1 or 0.9 with probability $1/2$ each, and player 2's valuation equalling the product of his signal and player 1's valuation. As p increases from $1/2$ to 1, player 2's payoff increases monotonically from $1/40$ to $1/20$.

private values, one player is known to be stronger than the other, and only one player has private information.

4.1.1 Private Values, Only the Strong Player Is Informed

Suppose that players have private values, player 2 has no private information (so he only has one type, s_2), and player 1 is "stronger," in that his valuation for the prize is always higher than that of player 2. Without loss of generality, each player's type equals his valuation for the prize, and $\succ_1$ equals $>$. That player 1 is stronger means that $s_1 \geq s_2$ for any type s_1 of player 1.

The equilibrium can be described in closed form. The number of intervals in the joint partition is n_1 (the cardinality of S_1), and the equilibrium bidding range is $[0, s_2]$. The equilibrium densities are

$$g_1\left(s_1^j, x\right) = \begin{cases} \frac{1}{f\left(s_1^j\right)s_2} & \text{if } x \text{ is in } \left[s_2 \sum_{k=j+1}^{n_1} f\left(s_1^k\right), s_2 \sum_{k=j}^{n_1} f\left(s_1^k\right)\right] \\ 0 & \text{otherwise} \end{cases}$$

and

$$g_2\left(s_2, x\right) = \begin{cases} \frac{1}{s_1^j} & \text{if } x \text{ is in } \left[s_2 \sum_{k=j+1}^{n_1} f\left(s_1^k\right), s_2 \sum_{k=j}^{n_1} f\left(s_1^k\right)\right] \\ 0 & \text{otherwise} \end{cases}.$$

In addition, player 2 chooses 0 with probability $1 - s_2 \sum_{k=1}^{n_1} f\left(s_1^k\right)/s_1^k \geq 0$.[21]

Compare this equilibrium to the one of the complete-information all-pay auction in which player 2's valuation is s_2 and player 1's valuation is s_1^j for some $j \leq n_1$. In the complete-information contest, player 1 mixes uniformly on $[0, s_2]$ with density $1/s_2$, and player 2 mixes uniformly on $[0, s_2]$ with density $1/s_1^j$ and bids 0 with probability $1 - s_2/s_1^j$. In both contests, players choose bids from $[0, s_2]$, player 1's unconditional bid distribution is the same (it is uniform with density $1/s_2$), and player 2's payoff is 0. Denote by Δ_1^j

[21] The inequality follows from

$$s_2 \sum_{k=1}^{n_1} \frac{f\left(s_1^k\right)}{s_1^k} \leq \frac{s_2}{s_1^{n_1}} \sum_{k=1}^{n_1} f\left(s_1^k\right) = \frac{s_2}{s_1^{n_1}} \leq 1.$$

If player 1 has at least two types (so the first inequality is strict) or $s_1^n > s_2$ (so the second inequality is strictly), then the atom is of positive measure. (Equivalently, if player 1 has a type higher than s_2.)

the difference between player 1's payoff in the incomplete-information contest when his valuation is s_1^j, and his payoff in the complete-information contest when his valuation is s_1^j. This difference is non-negative, because by bidding s_2 in the incomplete-information contest player 1 can obtain $s_1^j - s_2$, which is his payoff in the complete-information contest. Moreover, $\Delta_1^1 = 0$, because s_2 is a best response for player 1 in the incomplete-information contest when his valuation is s_1^1. But Δ_1^j increases in j and, in particular, is positive for $j > 1$. To see why, denote by $\bar{s}_1^j = s_2 \sum_{k=j}^{n_1} f\left(s_1^k\right)$ the upper bound of the bidding interval of player 1's type s_1^j in the incomplete-information contest. By bidding $\bar{s}_1^j$ in the incomplete-information contest, player 1 wins with probability $1 - s_2 \sum_{k=1}^{j-1} f\left(s_1^k\right)/s_1^k$. By bidding $\bar{s}_1^j$ in the complete-information contest, player 1 wins with probability

$$1 - \frac{s_2}{s_1^j} + \frac{\bar{s}_1^j}{s_1^j} = 1 - \frac{s_2 - \bar{s}_1^j}{s_1^j} = 1 - \frac{s_2\left(1 - \sum_{k=j}^{n_1} f\left(s_1^k\right)\right)}{s_1^j} = 1 - \frac{s_2 \sum_{k=1}^{j-1} f\left(s_1^k\right)}{s_1^j}.$$

The difference between these probabilities is

$$1 - s_2 \sum_{k=1}^{j-1} \frac{f\left(s_1^k\right)}{s_1^k} - \left(1 - s_2 \sum_{k=1}^{j-1} \frac{f\left(s_1^k\right)}{s_1^j}\right) = s_2 \sum_{k=1}^{j-1} f\left(s_1^k\right)\left(\frac{1}{s_1^j} - \frac{1}{s_1^k}\right),$$

and this difference multiplied by s_1^j equals Δ_1^j, so

$$\Delta_1^j = s_2 \sum_{k=1}^{j-1} f\left(s_1^k\right)\left(1 - \frac{s_1^j}{s_1^k}\right). \tag{9}$$

The right-hand side of (9) increases in j (because s_1^j decreases in j), so the increase in payoff relative to the complete-information contest is higher for lower types of player 1. This increase in payoff can be interpreted as the information rent that type s_1^j of player 1 obtains in excess of the "economic rent" that accrues to him because of his higher valuation. Figure 3 depicts the unique equilibrium when player 1's valuation is 3 or 5 with equal probability, and player 2's valuation is 2.

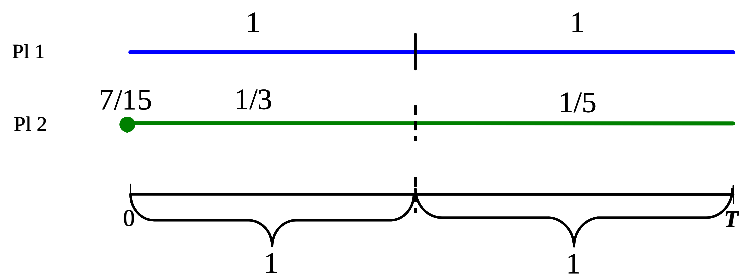

Figure 3: Equilibrium densities and player 2's atom

The equilibrium bidding range is $[0, 2]$, and player 2's payoff is 0, just like in the complete-information contests in which player 1's valuation is 3 or 5. Player 1's payoff when his valuation is 5 is 3, just like in the corresponding complete-information contest, but his payoff when his valuation is 3 is 7/5, higher than his payoff of 1 in the corresponding complete-information contest.

Consider the effect on player 2's equilibrium strategy of shifting probability mass from type s_1^j to type s_1^{j-1}, for some $j > 1$. The only change in the joint partition is that $\bar{s}_1^j$ is lowered to some $\tilde{s}_1^j$ in $\left[\bar{s}_1^{j+1}, \bar{s}_1^j\right)$ (where $\bar{s}_1^{n_1+1} = 0$), so the density of player 2's equilibrium strategy on $\left(0, \tilde{s}_1^j\right) \cup \left(\bar{s}_1^j, s_2\right]$ is not affected, and is lowered from $1/s_1^j$ to $1/s_1^{j-1}$ on $\left[\tilde{s}_1^j, \bar{s}_1^j\right)$. Because player 2's CDF still reaches 1 at s_2, we have

$$\tilde{G}_2(x) > G_2(x) \text{ for } x < \bar{s}_1^j \text{ and } \tilde{G}_2(x) = G_2(x) \text{ for } x \geq \bar{s}_1^j, \tag{10}$$

where $\tilde{G}_2$ is player 2's new equilibrium strategy. Therefore, the payoff of every type $k \geq j$ of player 1 increases, and that of every type $k < j$ does not change, which implies that player 1's expected payoff increases. Player 1's unconditional bid distribution remains uniform with density $1/s_2$ on $[0, s_2]$, and player 2's payoff remains 0. By definition, (10) implies that $\tilde{G}_2$ is FOSD by G_2. More generally, if player 1's signal distribution f is replaced by a distribution $\tilde{f}$ that FOSD f and has the same support, then player 2's new equilibrium strategy, $\tilde{G}_2$, is FOSD by G_2 (and player 1's expected payoff increases). This is because $\tilde{f}$ can be obtained from f in $n_1 - 1$ steps, by sequentially shifting probability mass $\sum_{k=j}^{n_1} \left(f\left(s_1^k\right) - \tilde{f}\left(s_1^k\right)\right) \geq 0$ from type s_1^j to type s_1^{j-1}, for $j = n_1, n_1 - 1, \ldots, 2$, so that each of the $n_1 - 1$ resulting CDFs in the sequence FOSD the previous one. A similar argument shows that $\tilde{G}_2$ is FOSD by G_2 even if $\tilde{f}$ and f do not have the same support. To see this, apply the sequential shifting procedure to the union of the supports of f and $\tilde{f}$, and note that the equilibrium is continuous in the distribution, so the property of FOSD is maintained when the probability of a type drops to 0. That $\tilde{G}_2$ is FOSD by G_2 implies that player 1's expected payoff increases.

4.1.2 Private Values, Only the Weak Player Is Informed

Suppose that players have private values, player 2 has no private information (so he only has one type, s_2), and player 1 is "weaker," in that his valuation for the prize is always lower than that of player 2. Without loss of generality, each player's type equals his valuation for the prize, and $\succ_1$ equals $>$. That player 1 is weaker means that $s_1 \leq s_2$ for any type s_1 of player 1.

The equilibrium can be described in closed form. Suppose that when the equilibrium is constructed player 1 exhausts his probability mass first. This implies that when player 1's valuation is s_1^j, he chooses a bid from an interval of length $f\left(s_1^j\right) s_2$ according to a uniform distribution with density $1/f\left(s_1^j\right) s_2$. On this interval, player 2 chooses a bid according to a uniform distribution with density $1/s_1^j$. Because $s_2 \sum_{k=1}^{n_1} f\left(s_1^k\right) = s_2$, the equilibrium bidding range would be $[0, s_2]$, on which player 2 would expend mass $s_2 \sum_{k=1}^{n_1} f\left(s_1^k\right)/s_1^k \geq 1$, with equality only if player 1 has one type and this type is s_2.[22] Therefore, player 2 exhausts his probability mass before player 1 does (so player 2 does not have an atom at 0). The equilibrium bidding range is determined by the type of player 1 in whose bidding interval player 2 exhausts his probability mass. This is type m, which is given by

$$m = 1 + \max\left\{ j : s_2 \sum_{k=1}^{j} \frac{f\left(s_1^k\right)}{s_1^k} < 1 \right\}.$$

Every type s_1^j, $j < m$, of player 1 exhausts his probability mass on an interval of length $f\left(s_1^j\right) s_2$, as described above. On these intervals player 2 expends mass $s_2 \sum_{k=1}^{m-1} f\left(s_1^k\right)/s_1^k <$

[22] The inequality follows from

$$s_2 \sum_{k=1}^{n_1} \frac{f\left(s_1^k\right)}{s_1^k} \geq \frac{s_2}{s_1^1} \sum_{k=1}^{n_1} f\left(s_1^k\right) = \frac{s_2}{s_2^1} \geq 1.$$

If player 1 has at least two types (so the first inequality is strict) or $s_1^1 < s_1$ (so the second inequality is strictly), then the inequality is strict. (Equivalently, if player 1 has a type lower than s_2.)

1. Let $\mu = 1 - s_2 \sum_{k=1}^{m-1} f\left(s_1^k\right)/s_1^k$. Type s_1^m chooses a bid from the interval on which player 2 exhausts his remaining mass of μ. Player 2 chooses a bid from this interval according to a uniform distribution with density $1/s_1^m$, so the length of the interval is μs_1^m. Therefore, the equilibrium bidding range is $\left[0, \mu s_1^m + s_2 \sum_{k=1}^{m-1} f\left(s_1^k\right)\right]$. The equilibrium densities are

$$g_1\left(s_1^j, x\right) = \begin{cases} \frac{1}{f\left(s_1^m\right)s_2} & \text{if } j = m \text{ and } x \text{ is in } \left[0, \mu s_1^m\right] \\ \frac{1}{f\left(s_1^j\right)s_2} & \text{if } x \text{ is in } \left[\mu s_1^m + s_2 \sum_{k=j+1}^{m-1} f\left(s_1^k\right), \mu s_1^m + s_2 \sum_{k=j}^{m-1} f\left(s_1^k\right)\right] \\ 0 & \text{otherwise} \end{cases}$$

and

$$g_2\left(s_2, x\right) = \begin{cases} \frac{1}{s_1^m} & \text{if } x \text{ is in } \left[0, \mu s_1^m\right] \\ \frac{1}{s_1^j} & \text{if } x \text{ is in } \left[\mu s_1^m + s_2 \sum_{k=j+1}^{m-1} f\left(s_1^k\right), \mu s_1^m + s_2 \sum_{k=j}^{m-1} f\left(s_1^k\right)\right] \\ 0 & \text{otherwise} \end{cases} .$$

In addition, type m of player 1 chooses 0 with probability $1 - \mu s_1^m / f\left(s_1^m\right) s_2$, and types $j > m$ of player 1 bid 0.

Compare this equilibrium to the one of the complete-information all-pay auction in which player 2's valuation is s_2 and player 1's valuation is s_1^j for some $j \leq n_1$. In the complete-information contest, player 2 mixes uniformly on $\left[0, s_1^j\right]$ with density $1/s_1^j$, and player 1 mixes uniformly on $\left[0, s_1^j\right]$ with density $1/s_2$ and bids 0 with probability $1 - s_1^j/s_2$. In both contests, player 1's unconditional bid distribution above 0 is the same (it is uniform with density $1/s_2$). But in the incomplete-information contest the equilibrium bidding range is $\left[0, \mu s_1^m + s_2 \sum_{k=1}^{m-1} f\left(s_1^k\right)\right]$, and the upper bound of this range is in $\left(s_1^{n_1}, s_1^1\right]$ (because the density of player 2's bid distribution is $1/s_1^1$ on some interval, and may be higher elsewhere, but nowhere higher than $1/s_1^{n_1}$). Therefore, player 2's payoff in the incomplete information contest is at least as high as in the complete-information contest in which $s_1^j = s_1^1$, but is lower than in the complete-information contest in which $s_1^j = s_1^{n_1}$. Player 1's payoff in the complete-information contest is 0. Denote by Δ_1^j player 1's payoff in the incomplete-information contest when he observes signal s_1^j. Clearly, $\Delta_1^j = 0$ for $j \geq m$. Moreover, Δ_1^j decreases in j for $j \leq m$ and, in particular, is positive for $j < m$. This is because by bidding the top of type s_1^j's bidding interval, type s_1^{j-1} obtains a higher payoff than type s_1^j does (he wins with the same probability, but his valuation for the prize is higher). This increase in payoff can be interpreted as the information rent that player 1 obtains in excess of his "economic rent" of 0. In contrast to Section 4.1.1, where lower types had higher information rents, here higher types have only weakly higher information rents: only types s_1^j, $j < m$, have positive payoffs. For a type to have positive information rents, the probability of lower types has to be sufficiently high.

Figure 4 demonstrates this by considering two contests, which differ in the probability that player 1's type is low.

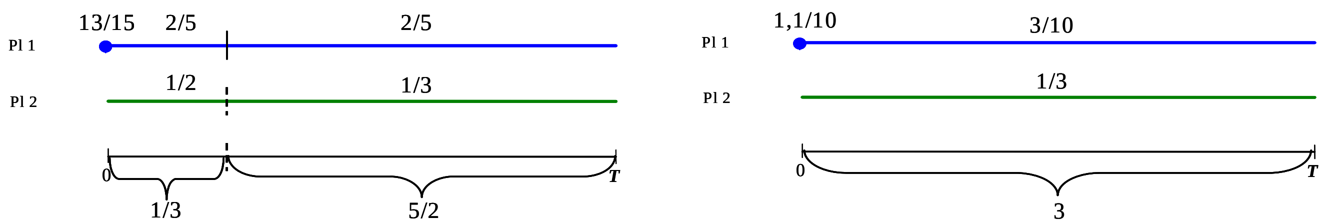

Figure 4: Equilibrium densities and player 1's atom

The left-hand side of Figure 4 depicts the unique equilibrium when player 1's valuation is 2 or 3 with equal probability, and player 2's valuation is 5. The equilibrium bidding range is $[0, 17/6]$, larger than that of the complete-information contest in which player 1's valuation is 2, $[0, 2]$, and smaller than that of the complete-information contest in which player 1's valuation is 3, $[0, 3]$. Player 1's payoff when his valuation is 2 is 0, just like in the complete-information contest, but his payoff when his valuation is 3 is $1/6$, higher than his payoff of 0 in the complete-information contest, so his expected payoff is $1/12$. This is because the probability that player 1's valuation is low is high enough for his high type to obtain a positive information rent. Player 2's payoff is $13/6$, higher than his payoff in the complete-information contest in which player 1's valuation is 3, and lower than his payoff in the complete-information contest in which player 1's valuation is 2. The right-hand side of Figure 4 depicts the unique equilibrium when player 1's valuation is 2 with probability $1/3$ and 3 with probability $2/3$, and player 2's valuation is 5. The equilibrium bidding range is $[0, 3]$, just like in the complete-information contest in which player 1's valuation is 3. Player 1's payoff is 0 regardless of his valuation. This is because the probability that player 1's valuation is low is not high enough for his high type to obtain a positive information rent. Player 2's payoff is 2, just like in the complete-information contest in which player 1's valuation is 3.

Consider the effect of shifting probability mass from type s_1^j to type s_1^{j-1}, for some $j > 1$. If $j > m$, then the equilibrium does not change, and the payoff of every type of each player remains the same.[23] If $j \leq m$, then the length of the interval on which type s_1^{j-1} bids increases, and the length of the interval on which type s_1^j decreases by the same amount. This implies that player 2 expends less probability mass on the original equilibrium bidding range, because player 2's bidding density is lower on the interval on which type s_1^{j-1} bids than on the interval on which type s_2^j bids ($1/s_1^{j-1}$ versus $1/s_2^j$). As a result, the equilibrium bidding range increases, and player 1's atom at 0 decreases (but the density of his unconditional bid distribution above 0 remains $1/s_2$), so player 2's payoff decreases. The same is true for any FOSD shift of player 1's signal distribution. In contrast, player 1's payoff may increase or decrease: the increase in the bidding range lowers the payoff for some of his types, but the increase in the probability of type s_1^{j-1}, whose payoff is higher than that of type s_1^j, increases his expected payoff.[24]

4.2 Common Values

Suppose that the value of the prize is common to both players, and denote the common valuation function by $V : S_1 \times S_2 \to \mathbb{R}_{++}$. In equilibrium, the unconditional distribution of players' bids is the same, regardless of the information structure and the function V. To see why, note that for almost any x in $(0, T]$ we have

$$g_i(x) = f(s_i(x))\, g_i(s_i(x), x) = \frac{f(s_i(x))}{f(s_i(x)\,|s_{-i}(x))\, V(s_1(x), s_2(x))} \tag{11}$$

$$= \frac{f(s_1(x))\, f(s_2(x))}{f(s_1(x), s_2(x))\, V(s_1(x), s_2(x))},$$

where g_i is the density of G_i, and $s_i(x)$ is the signal of player i for which x is a best response. Because the right-hand side of (11) is independent of i, players' densities are

[23] If $j = m + 1$, the density with which type s_1^m chooses positive bids decreases, but the increase in $f(s_1^m)$ precisely compensate for this, so that player 1's unconditional bid distribution does not change, and neither do the intervals on which his various types bid.

[24] In the example of Figure 4, when the probability p that player 1's valuation is 3 increases, player 1's payoff first increases ($p \leq 3/10$) and then decreases ($p > 3/10$).

equal for almost any x in $(0,T]$. In particular, both players exhaust the same probability mass on $(0,T]$. And since at most one player has an atom at 0, no player has an atom at 0. Therefore, the lowest type of each player has a payoff of 0. Other types' payoffs, however, and therefore players' expected payoffs, may differ between the players.

That players' strategies are identical from an ex-ante perspective is reminiscent of Engelbrecht-Wiggans, Milgrom, and Weber's (1983) result, who showed that this property holds in the equilibrium of a common-value first-price auction in which only one bidder is informed about the value of the object.

4.2.1 Common Values, One Informed Player

Suppose that the value of the prize is common to both players, that player 1 knows the common value, and that player 2 only knows it's distribution. This means that player 2 has only one type, s_2. Without loss of generality, player 1's type equals the common value, so $s_1^j = V_1\left(s_1^j, s_2\right) = V_2\left(s_2, s_1^j\right)$, and $\succ_1$ equals $>$. In this case, the equilibrium can be described in closed form. The number of intervals in the joint partition is n_1, and no player has an atom at 0. The equilibrium densities are

$$g_1\left(s_1^j, x\right) = \begin{cases} \frac{1}{f\left(s_1^j\right)s_1^j} & \text{if } x \text{ is in } \left[\sum_{k=j+1}^{n_1} f\left(s_1^k\right)s_1^k, \sum_{k=j}^{n_1} f\left(s_1^k\right)s_1^k\right] \\ 0 & \text{otherwise} \end{cases}$$

and

$$g_2\left(s_2, x\right) = \begin{cases} \frac{1}{s_1^j} & \text{if } x \text{ is in } \left[\sum_{k=j+1}^{n_1} f\left(s_1^k\right)s_1^k, \sum_{k=j}^{n_1} f\left(s_1^k\right)s_1^k\right] \\ 0 & \text{otherwise} \end{cases}.$$

Both players win the prize with the same probability, because their unconditional bid distributions are the same, as explained above. Player 2's payoff is 0, but player 1's expected payoff is positive if he has more than one type, because he places higher bids, and therefore wins more often, when the prize is more valuable.

Player 1's signal can also be interpreted as his expectation of the prize's value, so that some uncertainty regarding the prize's value remains after player 1 observes his signal. This makes it possible to investigate how the degree to which player 1 is informed affects players' equilibrium behavior. To do this requires comparing the informativeness of different probability distributions over player 1's signals. For two probability distributions $\tilde{f}$

and f with finite supports over the positive reals I say that $\tilde{f}$ is *more informative* than f if $\tilde{f}$ is a mean-preserving spread (MPS) of f. That is, if $\tilde{v} = v + x$, where $\tilde{v}$ and v are random variables whose distributions are $\tilde{f}$ and f, and x is a random variable whose expectation conditional on v is 0. For example, if $\tilde{f}(1) = \tilde{f}(3) = 1/2$ and $f(2) = 1$, then $\tilde{f}$ is more informative than f (x equals 1 and -1 with equal probability). In this case, $\tilde{f}$ could represent player 1 knowing the common value of the prize (1 or 3), and f could represent player 1 knowing only the expected value of the prize (2).

Proposition 3 *Suppose that $\tilde{f}$ is more informative than f. Then G FOSD $\tilde{G}$, where G and $\tilde{G}$ are players' unconditional equilibrium bid distributions under f and $\tilde{f}$.*

Intuitively, the more informed player 1 is, the less aggressive is players' bidding behavior. This implies the following result.

Corollary 4 *The more informed player 1 is, the higher his payoff is, and the lower are the overall expenditures.*

Proof. Since overall expenditures are the sum of players' expected bids, the second part of the corollary follows directly from FOSD. For the first part, note that the more informed player 1 is, the higher his probability of winning at any bid. This too follows directly from FOSD. ■

4.3 Quasi-Symmetric Players

A contest is quasi-symmetric if $S = S_1 = S_2$, $f(s_i|s_{-i}) = f(s_{-i}|s_i)$ for any $s_i = s_{-i}$ in S, and $V_1(s,s) = V_2(s,s)$ for every s in S.[25] In a quasi-symmetric contest the equilibrium is symmetric and can be described in closed form. The number of intervals in the joint partition equals the number of signals in S, n, and no player has an atom at 0. Let $V(s) = V_1(s,s) = V_2(s,s)$. The equilibrium density $g = g_1 = g_2$ is

$$g\left(s^j, x\right) = \begin{cases} \frac{1}{f(s^j|s^j)V(s^j)} & \text{if } x \text{ is in } \left[\sum_{k=j+1}^{n} f\left(s^k|s^k\right) V\left(s^k\right), \sum_{k=j}^{n} f\left(s^k|s^k\right) V\left(s^k\right)\right] \\ 0 & \text{otherwise} \end{cases}.$$

[25]For a quasi-symmetric contest to be symmetric, we must have that $f(s,s') = f(s',s)$ and $V_1(s,s') = V_2(s,s')$ for every s and s' in S.

The equilibrium is efficient, because higher types choose bids from higher intervals and the equilibrium is symmetric.

5 Equilibrium with a Reserve Price

Suppose that Condition M holds and a reserve price $r > 0$ is introduced. The reserve price corresponds to a minimum investment necessary to win the contest. A player who bids below r loses, regardless of what the other player bids. If r is high enough, then the players bid only 0 or r. This case is covered by Proposition 4 below. If r is not too high, then an equilibrium consists of two regions. Among the bids up to r, the players bid only 0 or r, and at most one player bids r. Above r, much of the previous analysis applies: for each player's type that bids above r, the set of best responses above r is an interval, these intervals are higher for higher types, and the union of these intervals for each player across his types is also an interval. Therefore, an equilibrium with a reserve price can be obtained by identifying a bid $b \leq r$, such that players' bidding behavior above r is a "right shift" of their bidding behavior above b in the equilibrium without a reserve price. Bids below b without a reserve price correspond to 0 or r with a reserve price. The case of a not-too-high reserve price is covered in Proposition 5 below. The bottom part of Figure 5 depicts an equilibrium structure consistent with the introduction of a not-too-high reserve price to the contest whose equilibrium structure is depicted in the top part of Figure 5.

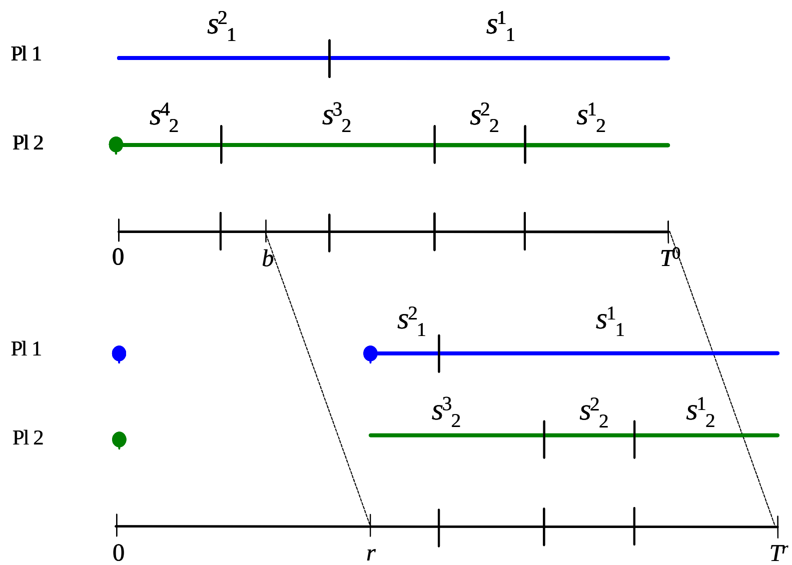

Figure 5: A possible equilibrium configuration of players' atoms and best response sets when player 1 has two signals and player 2 has four signals, without a reserve price (top), and with a reserve price (bottom)

The bid b is unique: it is the highest bid such that in the equilibrium without a reserve price, for at least one player, the gross winnings (excluding the bid payment) at that bid of the type for whom the bid is a best response are no higher than r. Indeed, if b were lower, then for at least one player the gross winning of the type for whom the bid is a best response without a reserve price would be lower than r, so he would not be willing to bid slightly above r, as required by the equilibrium structure (see Figure 5). If b were higher, then the gross winnings of the types of both players for whom the bid is a best response without a reserve price would be higher than r. But then neither would be willing to bid 0, which would imply they both have an atom at r, a contradiction. Even though b is unique (and easy to identify, as shown below), the mapping of bids lower than b may lead to multiple equilibria. These equilibria differ only in that some of the bids lower than b correspond to bidding 0 in one equilibrium and r in another equilibrium. All equilibria are, however, payoff equivalent.

I now describe players' equilibrium strategies in greater detail. For expositional simplicity, for the remainder of the section I assume that Condition M holds.[26] Denote by $\mathbf{G}^0 = (G_1^0, G_2^0)$ the unique equilibrium of the contest with a reserve price of 0, i.e., without a reserve price. For a bid x in $(0, T^0]$, where T^0 is the common supremum of players' best responses in $\mathbf{G}^0$, denote by $s_i(x)$ player i's (lowest) type for which x is a best response in $\mathbf{G}^0$.[27] Denote by

$$v_i^0(x) = \sum_{s_{-i} \in S_{-i}} f(s_{-i}|s_i(x)) V_i(s_i(x), s_{-i}) G_{-i}^0(s_{-i}, x)$$

player i's expected gross winnings without a reserve price if his type is $s_i(x)$, he bids x, and the other player plays G_{-i}^0. Note that $v_i^0(\cdot)$ increases on $(0, T^0]$,[28] and

$$v_i^0(T^0) = \sum_{s_{-i} \in S_{-i}} f(s_{-i}|s_i^1) V_i(s_i^1, s_{-i}).$$

Let

$$b_i = \max\left\{x \in \left(0, T^0\right] : v_i^0(x) \le r\right\}$$

if this set is non-empty, and $b_i = 0$ otherwise.[29] Note that b_i weakly increases in r, and $b_i = T^0$ if and only if $v_i^0(T^0) \le r$. In addition, $b_i \le r$.[30] Also, $b_1 > 0$ or $b_2 > 0$, because at least one player does not have an atom at 0. The following result characterizes the set of equilibria when r is large.

Proposition 4 *Suppose that $b_i = T^0$. Then, for every p in $[0, 1]$, the following pair of*

---

[26] Even if Condition M does not hold, Proposition 4 and Lemma 5 below still hold. The other results in this section apply to the set of candidate equilibria in which higher types have higher best responses above the reserve price. Condition WM is sufficient for each candidate equilibrium to be an equilibrium.

[27] The bid x is a best response for two types of player i only if x is an endpoint of the interval of bids chosen by some type of player i.

[28] $v_i^0(x)$ is piecewise differentiable with slope 1 wherever it is differentiable, and jumps upward wherever it is not differentiable.

[29] Because $s_i(\cdot)$ is left continuous, b_i is well defined.

[30] If $b_i > r$, then $r < T^0$ (because $b_i \le T^0$), so $v_i^0(b_i) \le r$ implies that $v_i^0(r) < r$ (because $v_i^0(\cdot)$ increases on $\left(0, T^0\right]$). But $v_i^0(x) - x \ge 0$ for any x in $\left(0, T^0\right]$, because x is a best response in $\mathbf{G}^0$ for type $s_i(x)$ of player i.

strategies is an equilibrium. Every type of player i bids 0. Type s_{-i} of player $-i$ bids 0 if

$$\sum_{s_i \in S_i} f\left(s_i | s_{-i}\right) V_{-i}\left(s_{-i}, s_i\right) - r < 0, \tag{12}$$

and bids r if the reverse inequality holds. If (12) holds with equality (which happens for at most one type s_{-i}), then type s_{-i} of player $-i$ bids 0 with probability p, and r with probability $1-p$. All these equilibria are payoff equivalent. Moreover, every equilibrium is such a pair of strategies for some i for which $b_i = T^0$.

Proposition 4 describes the set of equilibria when the reserve price is high, so $b = T^0$, where $b = \max\{b_1, b_2\} > 0$. I now turn to the case in which the reserve price is not too high, so $b < T^0$.

Lemma 5 *If $b < T^0$, then in any equilibrium the union of each player's best response sets across his types includes bids higher than r.*

Choose an equilibrium $\mathbf{G}^r = (G_1^r, G_2^r)$ of the contest with a reserve price. Denote by $BR_i^{r+}(s_i)$ player i's best responses higher than r when his type is s_i and the other player plays G_{-i}^r. Denote by S_i^{r+} the set of player i's types for which $BR_i^{r+}(s_i)$ is not empty. The next lemma shows that the set of players' best responses higher than r have a structure similar to that of the best response sets in the equilibrium of the contest without a reserve price.

Lemma 6 *Suppose that $b < T^0$. For $i = 1, 2$ and any s_i in S_i^{r+}, $BR_i^{r+}(s_i)$ is an interval. Also, if s_i is in S_i^{r+} and $s_i' \succ_i s_i$, then all of player i's best responses when his type is s_i' are higher than r. For any two consecutive signals $s_i' \succ_i s_i$ in S_i^{r+}, the upper bound of $BR_i^{r+}(s_i)$ is equal to the lower bound of $BR_i^{r+}(s_i')$. Moreover,*

$$\begin{aligned} &\sup \cup_{s_1 \in S_1^{r+}} BR_1^{r+}(s_1) = \sup \cup_{s_2 \in S_2^{r+}} BR_2^{r+}(s_2) \text{ and} \\ &\inf \cup_{s_1 \in S_1^{r+}} BR_1^{r+}(s_1) = \inf \cup_{s_2 \in S_2^{r+}} BR_2^{r+}(s_2) = r. \end{aligned} \tag{13}$$

Denote by T^r the common supremum in (13). Lemma 6 shows that the construction procedure described in Section 3 applies to bids in $(r, T^r]$. Therefore, above r any equilibrium coincides with the equilibrium without a reserve price starting from some point. The next lemma shows that this point is b, as depicted in Figure 5.

Lemma 7 *For $i = 1, 2$, any signal s_i, and every $x \geq 0$, we have*

$$G_i^r(s_i, r + x) = G_i^0(s_i, b + x). \tag{14}$$

Lemma 7 pins down $\mathbf{G}^r$ above r. To completely characterize the set of equilibria, it remains to specify how players choose bids from $\{0, r\}$, as in Proposition 4.

Proposition 5 *Suppose that $b < T^0$, and $b_i = b$ for player i. Then, for every p in $[0, 1]$, the following pair of strategies is an equilibrium. Every type $s_i \prec_i s_i(b)$ of player i bids 0, and type $s_i(b)$ of player i bids 0 with probability $G_i^0(s_i(b), b)$. Every type $s_i \succeq_i s_i(b)$ of player i chooses bids higher than r according to (14). Type $s_{-i} \prec_i s_{-i}(b)$ of player $-i$ bids 0 if*

$$\sum_{s_i \in S_i} f(s_i|s_{-i}) V_{-i}(s_{-i}, s_i) G_i^0(s_i, b) - r < 0, \tag{15}$$

and bids r if the reverse inequality holds. If (15) holds with equality (which happens for at most one type s_{-i}), then type s_{-i} of player $-i$ bids 0 with probability p, and r with probability $1 - p$. Type $s_{-i}(b)$ of player $-i$ chooses with probability $G_i^0(s_i(b), b)$ bids from $\{0, r\}$ according to (15), as specified above for lower types. Every type $s_{-i} \succeq_{-i} s_{-i}(b)$ of player $-i$ chooses bids higher than r according to (14) (with $-i$ in place of i). All these equilibria are payoff equivalent. Moreover, every equilibrium is such a pair of strategies for some i for which $b_i = b$.

Propositions 4 and 5 imply that introducing a reserve price makes both players weakly worse off. This is because $b_i \leq r$, as stated above, so $b \leq r$, which means that above r players' strategies are a "right shift" of their strategies above b without a reserve price, as depicted in Figure 5. Thus, players face tougher competition with a reserve price, which lowers their payoffs. The following result generalizes this observation.

Corollary 5 *The equilibrium payoff of every type of every player weakly decreases in the reserve price.*

Propositions 4 and 5 show that multiple equilibria may exist. This occurs when (12) or (15) hold with equality, which happens for at most one type of each player (because of Condition M). The equilibria differ only in the probabilities with which that particular type bids 0 and r. Therefore, when the probability of each type is small (so the number of types is large), the difference between any two equilibria is small.[31] This observation is consistent with Lizzeri and Persico's (2000) result, which implies that with a continuum of types, each of which occurs with probability 0, and a sufficiently high reserve price there is a unique equilibrium.[32] Their result does not apply when there is no reserve price, or when the reserve price is low. In contrast, Propositions 4 and 5 characterize the set of equilibria for any reserve price. Appendix C contains some examples of contests with a reserve price and their equilibria.

6 Conclusion

This paper has investigated an asymmetric two-player all-pay auction. The novel features are a finite number of signals for each player, an asymmetric signal distribution and interdependent valuations, and a non-restricted reserve price. The constructive characterization of the set of equilibria has shown that under a monotonicity condition there is a unique equilibrium without a reserve price, and with a reserve price all equilibria are payoff equivalent and differ in the behavior of at most one type for each player. A closed-form equilibrium characterization and comparative statics have been given for some special cases.

One direction for future research is to apply the equilibrium construction results to additional special cases in order to derive comparative statics and closed-form equilibrium characterizations. These can be used to investigate models of real-world competitions, such as the research and development setting described in the Introduction. The model might be particularly useful for the analysis of examples and applications of contests in which

[31]For example, the distance between any two equilibria is small according to the metric induced by the sup norm.

[32]The reserve price must be high enough to exclude, for each bidder, a positive measure of types from bidding, regardless of what the other bidder does.

incomplete information is naturally modeled by a finite number of signals. Another research direction is to extend the model and results to more than two players and additional signal distributions. This seems to be a non-trivial task, because much of the equilibrium analysis is driven by these assumptions.

A Proofs

A.1 Proof of Lemma 1

Suppose that $x > y$. Because x is in $BR_i(s_i)$, player $-i$ can only choose x with positive probability if ties at x are decided in favor of player i, and similarly for y. Therefore,

$$u_i(s_i, x) - u_i(s_i, y) = \sum_{s_{-i} \in S_{-i}} (f(s_{-i}|s_i) V_i(s_i, s_{-i}) (G_{-i}(s_{-i}, x) - G_{-i}(s_{-i}, y))) - (x - y) \geq 0, \tag{16}$$

where $u_i(s_i, z)$ is player i's expected payoff when he observes signal s_i and bids z, and the last inequality follows from $u_i(s_i, x) \geq u_i(s_i, y)$, because x is in $BR_i(s_i)$. This last inequality and $x > y$ imply that $G_{-i}(s_{-i}, x) - G_{-i}(s_{-i}, y) > 0$ for at least one signal s_{-i}; and for every signal s_{-i} we have $G_{-i}(s_{-i}, x) - G_{-i}(s_{-i}, y) \geq 0$ because $G_{-i}(s_{-i}, \cdot)$ is a CDF and $x > y$. Therefore, Condition M implies that the value of the left-hand side of (16) increases if s_i is replaced with s'_i. This shows that $u_i(s'_i, x) > u_i(s'_i, y)$, so y is not in $BR(s'_i)$, a contradiction. Therefore, $x \leq y$.

A.2 Proof of Lemma 2

For (i), suppose that type s_1 of player 1 and type s_2 of player 2 chose x with positive probability. Because $f(s_1|s_2)$ is positive, player 2 could do strictly better by choosing a bid slightly above x, so x cannot be a best response for type s_2 of player 2, a contradiction. For (ii), suppose that type s_i of player i chose a positive x with positive probability. Similarly to (i), no type of the other player would have best responses on some positive-length interval with upper bound x. But then player i could do strictly better by bidding slightly below x, so x cannot be a best response for his type s_i, a contradiction. For (iii), note that (ii) proved that each player's CDF is continuous above 0 for any of his types. Therefore, if a positive x is not a best response for any type of player i, the same is true for all bids in a some maximal neighborhood of x. This implies that the other player also does not choose any bids in this neighborhood. But then, again by continuity, no player would have a best response at the top of this neighborhood (only an atom at the top of the neighborhood could make the other player willing to bid there), so this neighborhood is unbounded. For (iv), suppose that 0 is not a best response for one of the players and that player does not have best responses arbitrarily close to 0. This means that the player does not have best responses in some interval with lower endpoint 0. By (iii), the player does not have any best-responses, so we do not have an equilibrium.

A.3 Proof of Lemma 3

By Lemma 1 and part (iii) of Lemma 2, $BR_i(s_i)$ is an interval. By part (iii) of Lemma 2, $BR_i(s_i) \cap BR_i(s'_i)$ is not empty, and because the equilibrium is monotonic, this intersection can include only the boundaries of the best-response sets. Parts (iii) and (iv) of Lemma 2 imply (3).

A.4 Proof of Proposition 1

It suffices to show that every type of each player chooses best responses with probability 1. Denote by $0 = l_{n_1}, t_{n_1}, l_{n_1-1}, t_{n_1-1}, \ldots, l_2, t_2, l_1, t_1 = T$ the partition of $[0, T]$ identified by the procedure for player 1, so his type s_1^k chooses bids from the interval (l_k, t_k) (that is, $g_1\left(s_1^k, \cdot\right) > 0$ on this interval). Note that $t_k = l_{k-1}$ for any $k > 1$. Suppose that player 1's type is s_1^k. By construction, player 2's strategy is continuous at all positive bids, and player 1 obtains the same payoff at every bid $(l_k, t_k]$. Moreover, if player 2 does not have an atom at l_k, then player 1 obtains the same payoff at l_k as well. If player 2 does have an atom at l_k, then $l_k = 0$ is not a profitable deviation for player 1. Therefore, to show that type player 1 does not have profitable deviations, it suffices to rule out bids lower than l_k or higher than t_k. Suppose that player 1 has a profitable deviation lower than l_k, and let $[l_j, t_j]$ be the highest interval below $[l_k, t_k]$ that contains a profitable deviation y. Because $t_j = l_{j-1}$, $y < t_j$. By construction, player 1 obtains the same payoff at y and t_j when his type is s_1^j.[33] Therefore, because $s_1^k \succ_i s_1^j$, Condition WM implies that player 1 obtains weakly more at $t_j = l_{j-1}$ than at y when his type is s_1^k (this follows from (16) and an argument similar to the one that follows (16) in the proof of Lemma 1). If $j - 1 = k$, this shows that y is not a profitable deviation. If $j - 1 > k$, then $[l_{j-1}, t_{j-1}]$ contains a profitable deviation, contradicting the definition of $[l_j, t_j]$ as the highest interval below $[l_k, t_k]$ that contains a profitable deviation. This shows that there are no profitable deviations below $[l_k, t_k]$. A similar argument shows that there are no profitable deviations in $(t_k, T]$, by considering the lowest interval above $[l_k, t_k]$ that contains a hypothesized profitable deviation. Bids higher than T are strictly dominated by T. Therefore, player 1 does not have profitable deviations. The same argument shows that player 2 also chooses best responses with probability 1.

A.5 Proof of Proposition 3

Because $\tilde{v} = v + x$ and $\tilde{f}$ and f have finite supports, so does x conditional on v. Denote by x_s the random variable induced by x conditional of the realization s of v. Because FOSD is a transitive relation and f has finite support, it suffices to prove the proposition for a random variable x for which $x_s \equiv 0$ for all but one realization s of x. Therefore, consider $\tilde{f}$ for which $\tilde{v}$ satisfies $\tilde{v} = \begin{cases} v & \text{if } v \neq s^l \\ s^l + y & \text{if } v = s^l \end{cases}$, where s^l is some realization of v and y is a random variable with finite support and mean 0 that is statistically independent of v. Denote by $\tilde{s}^1 > \cdots > \tilde{s}^m$ the values that $\tilde{v}$ takes conditional on $v = s^l$, so $\sum_{j=1}^m \tilde{f}\left(\tilde{s}^j\right)\tilde{s}^j = f\left(s^l\right)s^l$ (and $\sum_{j=1}^m \tilde{f}\left(\tilde{s}^j\right) = f\left(s^l\right)$). Intuitively, $\tilde{v}$ conditional on $v = s^l$ "splits" s^l into a weighted

[33] If $y = 0$ and player 2 has an atom at 0, then choose a slightly higher y as the profitable deviation.

average of $\tilde{s}^1, \ldots, \tilde{s}^m$, where the weight of $\tilde{s}^j$ is $\tilde{f}\left(\tilde{s}^l\right)/f\left(s^l\right)$. I now describe how $\tilde{G}$ can be obtained from G by transitioning through a finite number of probability distributions such that each distribution is FOSD by the previous one. This will conclude the proof. Denote by $I = \left[a_l, a_l + f\left(s^l\right)s^l\right]$ the interval from which type s^l chooses bids under G. Modify G on I by replacing its density of $1/s^l$ there with those generated by $\tilde{s}^1, \ldots, \tilde{s}^m$, i.e., $1/\tilde{s}^k$ on $I^k = \left[a_l + \sum_{j=k+1}^{m} \tilde{f}\left(\tilde{s}^j\right)\tilde{s}^j, a_l + \sum_{j=k}^{m} \tilde{f}\left(\tilde{s}^j\right)\tilde{s}^j\right]$ for $k \leq m$. Denote the resulting function by G^0, and note that G^0 is a continuous probability distribution that coincides with G up to a_l and above $a_l + f\left(s^l\right)s^l$, and is weakly higher than G on I (because $\tilde{s}^m \leq s^j \leq \tilde{s}^1$ and G^0 coincides with G at a_l and $a_l + f\left(s^l\right)s^l$). Therefore, G FOSD G^0. Now, G and $\tilde{G}$ are piecewise linear and concave, as shown at the beginning of Section 4.1. G^0 is also piecewise linear, but need not be concave, because of its values on I. But the linear components in $\tilde{G}$ and G^0 are identical, so to obtain $\tilde{G}$ from G^0 it remains only to "shift" the densities on the intervals I^k, $k \leq m$, to their "correct" locations and obtain a concave function, which would necessarily be $\tilde{G}$. Because $\tilde{s}^1 \leq s^l$, begin by "moving I^1 downwards," immediately after the interval of bids from which type s^{j_1} chooses his bids under G, where s^{j_1} is the highest realization $s^j < s^l$ of v such that $s^j \leq \tilde{s}^1$. Denote the resulting probability distribution by G^1.[34] It is straightforward to verify that G^0 FOSD G^1. Continue in this way for $k = 2, \ldots, m$, "moving I^k downwards" if $\tilde{s}^k < s^l$ and "moving I^k upwards" if $\tilde{s}^k > s^l$ to obtain G^k. Then G^{k-1} FOSD G^k, and $G^m = \tilde{G}$.

A.6 Proposition 4

First, note that in any equilibrium both players choose bids only from $\{0, r\}$. Indeed, because $b_i = T^0$ implies that $v_i^0\left(T^0\right) \leq r$, and since $v_i^0\left(T^0\right)$ is the highest possible (gross) winnings for player i, he does not bid more than r. Therefore, player $-i$ does not bid more than r (for any such bid a slightly lower bid is better). Clearly, neither player chooses bids from $(0, r)$. To see that the proposed pairs of strategies are optimal, note that player i obtains at most 0 by bidding r, so bidding 0 is optimal for him. Therefore, player $-i$ wins with probability 1 by bidding r, so the left hand side of (12) describes his payoff when he bids r. This implies that the proposed strategies for player $-i$ are optimal and lead to the same payoffs. To see that every equilibrium is such a pair of strategies, note that in equilibrium at most one player bids r with positive probability (as in part (i) of Lemma 2). Therefore, in any equilibrium in which player $-i$ bids r with positive probability, every type of player i bids 0, and any such equilibrium is a pair of strategies as specified above. In any equilibrium in which player i bids r with positive probability, every type of player $-i$ bids 0, so $b_{-i} = T^0$ (otherwise bidding slightly above r would be a profitable deviation

[34] That is, set $G^1(x) = G^0(x)$ for bids x chosen under G by types s^j, where $s^j \leq s^{j_1}$ or $s^j > s^l$, set $G^1\left(x + \tilde{f}\left(\tilde{s}^1\right)\tilde{s}^1\right) = G^0(x) + \tilde{f}\left(\tilde{s}^1\right)$ for bids x chosen under G by types s^j, where $s^{j_1} < s^j < s^l$, and set the density of G^1 to be $1/\tilde{s}^1$ on an interval of length $\tilde{f}\left(\tilde{s}^1\right)\tilde{s}^1$ that begins at the highest bid chosen under G by type s^{j_1}.

for the highest type of player $-i$). The pairs of strategies described above, with $-i$ instead of i, describe all the equilibria in which every type of player $-i$ bids 0.

A.7 Proof of Lemma 5

If the claim is false, then in any equilibrium both players choose bids only from $\{0, r\}$, and at most one player chooses r with positive probability (as in part (i) of Lemma 2). Therefore, every type of some player i has a payoff of 0. But because $b_i < T^0$, we have $v_i^0 (T^0) > r$, so by bidding slightly above r player i's highest type can win with certainty and obtain a positive payoff, a contradiction.

A.8 Proof of Lemma 6

A proof similar to that of Lemma 2 shows that no player has atoms above r, at most one player has an atom at r, the union of each player's set of best responses higher than r across his types is an interval, and these intervals have the same upper bound and the same lower bound of r. A proof similar to that of Lemma 1 shows that for any two signals $s_i' \succ_i s_i$, such that s_i is in S_i^{r+}, and any x in $BR_i^{r+} (s_i)$ and y that is a best response for s_i', we have $y \geq x$. This implies the remainder of the claim, as in the proof of Lemma 3.

A.9 Proof of Lemma 7

Because the construction procedure described in Section 3 applies to bids in $(r, T^r]$, the statement of the lemma holds with some y in place of b in (14). Suppose that $y < b$, so that $y < b_i$ for some player i. Because $y < b_i$ and $v_i^0 (\cdot)$ increases on $(0, T^0]$, we have that $v_i^0 (y + \varepsilon) < r$ for small $\varepsilon > 0$. Consider type s_i of player i, who bids slightly above y in $\mathbf{G}^0$. By bidding slightly above r in $\mathbf{G}^r$, this type's (gross) winnings are less than r, so his payoff is negative. Therefore, $\mathbf{G}^r$ is not an equilibrium. Now suppose that $y > b$. Because $y > b_1$ and $y > b_2$, we have $v_1^0 (y) > r$ and $v_2^0 (y) > r$. Therefore, the payoffs in $\mathbf{G}^r$ of types $s_1 (y)$ and $s_2 (y)$ (the lowest types that bid y in $\mathbf{G}^0$) are positive. And because $G^r (s_1 (y), r) = G^0 (s_1 (y), y) > 0$ (where the inequality follows from $y > b > 0$ and the definition of $s_1 (y)$) and, similarly, $G^r (s_2 (y), r) > 0$, types $s_1 (y)$ and $s_2 (y)$ each have an atom at 0 and/or r. But because at most one player has an atom at r, either type $s_1 (y)$ or type $s_2 (y)$ (or both) have an atom at 0, leading to a payoff of 0 in $\mathbf{G}^r$, a contradiction.

A.10 Proof of Proposition 5

Similarly to the proofs of Propositions 1 and 4, it is straightforward to verify that the proposed pairs of strategies are equilibria. To see that every equilibrium is such a pair of strategies, recall that Lemma 7 pins down players' equilibrium behavior above r, and at most one player bids r with positive probability. Therefore, similarly to the proof of Proposition 4, any equilibrium in which player $-i$ bids r with positive probability is a pair of strategies as specified above. If there is an equilibrium in which player i bids r with positive probability, then $b_{-i} = b$. This is because $b_{-i} < b$ implies that type $s_i(b)$ of player 2 can obtain a positive payoff by bidding slightly above r, but because at most one player has an atom at r, type $s_i(b)$ of player $-i$ must bid 0 (and get 0) with probability $G_i^0(s_i(b), b) > 0$, a contradiction. Therefore all the equilibria in which player i bids r with positive probability are given by the pairs of strategies described above, with $-i$ instead of i.

A.11 Proof of Corollary 5

It suffices to show that the payoff of every type of every player at any given bid decreases in r when the other player plays his equilibrium strategy. For bids in $[0, r)$ this is true, because the payoff there is 0. For bids $x \geq r$, it suffices to show that $G_i^r(s_i, x)$ weakly decreases in r for every s_i in S_i and $i = 1, 2$, because this implies that the (gross) winnings at x for player $-i$ weakly decrease in r. Because the equilibrium above r is given by (14), it suffices to show that T^r weakly increases in r or, equivalently, that the increase in b resulting from an increase in r is no higher than the increase in r. But this follows from the definitions of $v_i^0(x)$ and b_i: $v_i^0(x)$ is piecewise differentiable with slope 1 wherever it is differentiable, and jumps upward wherever it is not differentiable, so an increase in r leads to a weakly lower increase in b_i, and therefore to a weakly lower increase in b.

B Equilibrium Ordering

The procedure for constructing the equilibrium candidate shows that players' types exhaust their probability masses in an order that depends on their valuation functions and the probability distribution. That is, the construction induces an ordering $(s^1, \ldots, s^{n_1+n_2})$ of the elements in $S_1 \cup S_2$, such that the probability mass associated with s^j is expended on an interval of bids whose lower bound is (weakly) lower than those of the intervals of bids that correspond to types $s^1, \ldots, s^{j-1}$. And if the last type in the ordering, $s^{n_1+n_2}$, is a type of player i, then the lower bound of the interval of bids of the last type of player $-i$ in the ordering is 0. The payoff of this last type of player $-i$ is 0, as is the payoff of all the types that appear later in the ordering (all of whom belong to player i). For example, the ordering that corresponds to Figure 1 is $(s_2^1, s_2^2, s_1^1, s_2^3, s_1^2, s_2^4)$, the lower bound of the interval of bids of type s_1^2 is 0, and the payoff of types s_1^2 and s_2^4 is 0.

In any such ordering, and for any pair of types of a player, the higher type appears before the lower type. Thus, the number of orderings of players' types that can be generated by varying players' valuation functions and the probability distribution is at most $(n_1 + n_2)!/(n_1!n_2!)$: this is the number of orderings of $n_1 + n_2$ elements, n_1 of which are identical and the other n_2 of which are identical. Conversely, it is easy to see that each ordering of n_1 identical elements and n_2 identical elements corresponds to an ordering of players' types for some valuation functions and probability distribution.[35] And which valuation functions and distributions correspond to which ordering is described by a set of inequalities, so for any fixed n_1 and n_2 the equilibrium candidate can be described in closed form.

For example, for $n_1 = 2$ and $n_2 = 1$ there are $(2+1)!/2!1! = 6/2 = 3$ possible orderings: (i) (s_1^1, s_1^2, s_2^1), (ii) (s_1^1, s_2^1, s_1^2), and (iii) (s_2^1, s_1^1, s_1^2). The configurations that correspond to the orderings are illustrated in Figure 6.

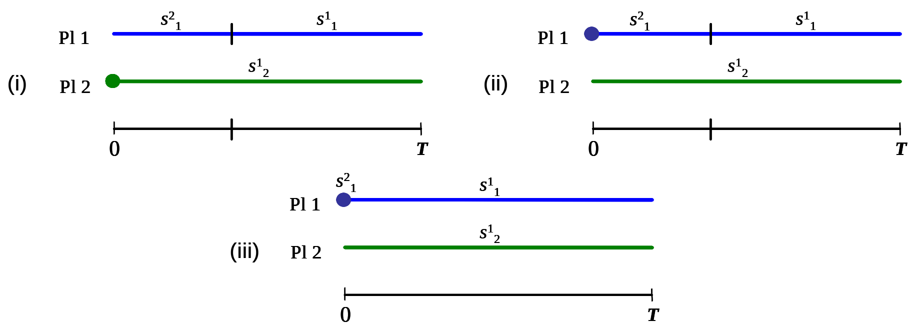

Figure 6: The possible configurations when player 1 observes one of two signals and player 2 has one signal

In (i), players' densities are $g_1(s_1^1, \cdot) = 1/f(s_1^1)V_2(s_2^1, s_1^1)$, $g_2(s_2^1, \cdot) = 1/V_1(s_1^1, s_2^1)$ in the top interval and $g_1(s_1^2, \cdot) = 1/f(s_1^2)V_2(s_2^1, s_1^2)$, $g_2(s_2^1, \cdot) = 1/V_1(s_1^2, s_2^1)$ in the bottom interval. Player 1 exhausts the probability mass associated with both his types before player 2 exhausts the mass associated with his single type, so

$$V_1\left(s_1^1, s_2^1\right) > f\left(s_1^1\right) V_2\left(s_2^1, s_1^1\right) \tag{17}$$

and

$$\left(1 - \frac{f(s_1^1)\,V_2(s_2^1, s_1^1)}{V_1(s_1^1, s_2^1)}\right) \underbrace{V_1\left(s_1^2, s_2^1\right)}_{\text{The reciprocal of player 2's density}} \geq \underbrace{f\left(s_1^2\right) V_2\left(s_2^1, s_1^2\right)}_{\text{The reciprocal of player 1's density}} . \tag{18}$$

Fixing the probability distribution and player 2's valuation function, this happens when $V_1(s_1^1, s_2^1)$ and $V_1(s_1^2, s_2^1)$ are large enough. In (ii), players' densities are as in (i). Player 2 exhausts the mass associated with his single type after player 1 exhausts the mass associated with his high type, so (17) holds, but before player 1 exhausts the mass associated with his low type, so the reverse of (18) holds. Fixing the probability distribution and player 2's valuation function, this happens when $V_1(s_1^1, s_2^1)$ is large enough and $V_1(s_1^2, s_2^1)$ is small enough. In (iii), players' densities in the single interval are as in the top interval in orderings (i) and (ii). Player 2 exhausts the mass associated with his single type before player 1

[35] If signal s_i of player i immediately follows signal s_{-i} of player $-i$ in the ordering (so the probability mass associated with s_{-i} is exhausted before that associated with s_i), then by increasing $V_i(s_i, s_{-i})$ the order of the two signals can be reversed.

exhausts the mass associated with his high type, so the reverse of (17) holds. Fixing player 1's probability distribution and player 2's valuation function, this happens when $V_1(s_1^1, s_2^1)$ is small enough.

For $n_1 = n_2 = 2$, there are $(2+2)!/(2!2!) = 24/4 = 6$ possible orderings: (i) $(s_1^1, s_1^2, s_2^1, s_2^2)$, (ii) $(s_1^1, s_2^1, s_1^2, s_2^2)$, (iii) $(s_1^1, s_2^1, s_2^2, s_1^2)$, (iv) $(s_2^1, s_2^2, s_1^1, s_1^2)$, (v) $(s_2^1, s_1^1, s_2^2, s_1^2)$, and (vi) $(s_2^1, s_1^1, s_1^2, s_2^2)$. The configurations that correspond to (i), (ii), and (iii) are illustrated in Figure 7 ((iv), (v), and (vi) are obtained from (i), (ii), and (iii) by switching the indices of players 1 and 2). The inequalities that determine which of the orderings obtains can be easily derived similarly to the case $n_1 = 2$ and $n_2 = 1$. Players' densities in the various intervals follow immediately from Figure 7.

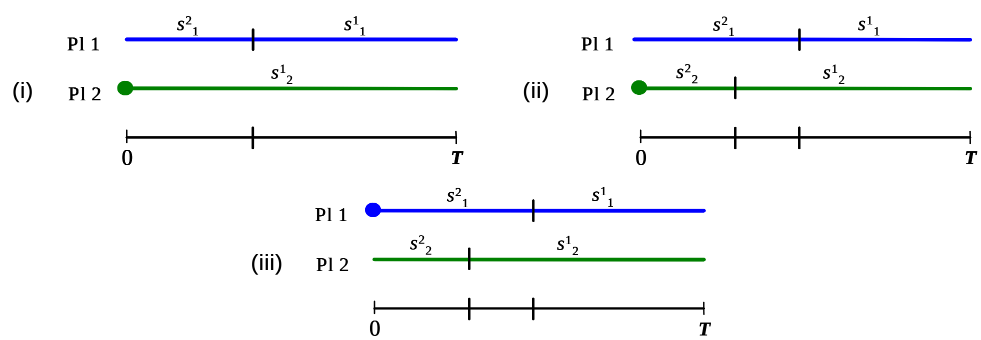

Figure 7: The equilibrium configurations that correspond to orderings (i), (ii), and (iii) when each player observes one of two signals

C Examples of Equilibrium with a Reserve Price

Consider a complete-information all-pay auction for a prize of common value V. Without a reserve price, $T^0 = V$ and each player mixes uniformly with density $1/V$ on $[0, V]$. Suppose a reserve price is introduced. Because $v_i^0(x) = x$ for $x \leq V$, we have $b_i = b = \min\{r, V\}$. If $r > V$, then both players bid 0 (in this case $b = V$, so Proposition 4 and (12) hold). If $r = V$, then one player bids 0 and the other player mixes between 0 and V (Proposition 4 holds and (12) holds with equality). If $r < V$, then on (r, V) both players mix uniformly with density $1/V$; one of the players bids 0 with his remaining probability, r/V, and the other player bids 0 with probability pr/V and V with probability $(1-p)r/V$, for some p in $[0, 1]$ (Proposition 5 holds and (15) holds with equality).

In a complete-information all-pay auction with asymmetric valuations, there is a unique equilibrium even with a reserve price, as long as the reserve price is not equal to the higher of the two players' valuations. To see this, denote by V_i player i's valuation for the prize, and let $V_1 > V_2$. Without a reserve price, $T^0 = V_2$, player 1 mixes uniformly with density $1/V_2$ on $[0, V_2]$, and player 2 chooses 0 with probability $(V_1 - V_2)/V_1$ and mixes uniformly with density $1/V_1$ on $(0, V_2)$. For the equilibrium with a reserve price, note that

$v_1^0(x) = V_1 - V_2 + x$ and $v_2^0(x) = x$ for $x \leq V_2$. Therefore, $b_1 = \max\{0, r - (V_1 - V_2)\}$ for $r < V_1$ and $b_1 = V_2$ for $r \geq V_1$, and $b_2 = r$ for $r < V_2$ and $b_2 = V_2$ for $r \geq V_2$. This implies that $b = b_2$. If $r > V_1$, then both players bid 0 (Proposition 4 and (12) hold). If $r = V_1$, then player 2 bids 0 and player 1 mixes between 0 and V_1 (Proposition 4 holds and (12) holds with equality for $i = 2$). If r is in $[V_2, V_1)$, then player 2 bids 0 and player 1 bids r (Proposition 4 holds and (12) holds with the reverse inequality for $i = 2$). If $r < V_2$, then on (r, V_2) both players mix uniformly with their respective densities, $1/V_2$ and $1/V_1$; player 1 bids r with his remaining probability, r/V_2, and player 2 bids 0 with his remaining probability, $(r + V_1 - V_2)/V_1$ (Proposition 5 holds and (15) holds with the reverse inequality for $i = 2$).

In contrast to the complete information case, when players have private information it may be that $b < r$, even when $b < T^0$, as depicted in Figure 5. To see this, consider a private value setting in which each player's valuation for the prize is 1 or 2 with equal probabilities. Without a reserve price, $T^0 = 3/2$, the low type of each player mixes uniformly with density 2 on $[0, 1/2]$, and the high type of each player mixes uniformly with density 1 on $[1/2, 3/2]$. This implies that $v_i^0(x) = \begin{cases} x & \text{if } x \leq 1/2 \\ x + 1/2 & \text{if } 1/2 < x \leq 3/2 \\ 2 & \text{if } x > 3/2 \end{cases}$. Therefore, for $r \leq 1/2$ we have $b = b_i = r$, as in the complete-information case. But for r in $[1/2, 1]$, we have $b = 1/2$, and for r in $(1, 2]$ we have $b = r - 1/2$.

References

[1] Amann, Erwin, and Wolfgang Leininger. 1996. "Asymmetric All-Pay Auctions with Incomplete Information: The Two-Player Case." *Games and Economic Behavior* 14, 1-18.

[2] Bulow, Jeremy I. and Levin, Jonathan. 2006. "Matching and Price Competition." *American Economic Review* 96, 652-68.

[3] Engelbrecht-Wiggans, Richard, Paul R. Milgrom, and Robert J. Weber. 1983. "Competitive Bidding and Proprietary Information." *Journal of Mathematical Economics* 11, 161-169.

[4] Hillman, Arye L., and John G. Riley. 1989. "Politically Contestable Rents and Transfers." *Economics and Politics* 1, 17-39.

[5] Konrad, Kai A. 2004. "Altruism and envy in contests: an evolutionary stable symbiosis." *Social Choice and Welfare* 22, 479-490.

[6] Konrad, Kai A. 2009. "Strategy and Dynamics in Contests." Oxford University Press, Oxford.

[7] Krishna, Vijay. 2002. "The Revenue Equivalence Principle," in *Auction Theory*, San Diego, California, Elsevier Science, pp. 31-2.

[8] Lizzeri, Alessandro, and Nicola Persico. 2000. "Uniqueness and Existence of Equilibrium in Auctions with a Reserve Price." *Games and Economic Behavior* 30, 83-114.

[9] Mares, Vlad, and Jeroen M. Swinkels. 2009. "On the Analysis of Asymmetric First Price Auctions." Mimeo.

[10] Morgan, John, and Vijay Krishna. 1997. "An Analysis of the War of Attrition and the All-Pay Auction." *Journal of Economic Theory*, 72, 343–362.

[11] Parreiras, Sergio, and Anna Rubinchik. 2009. "Contests with Many Heterogeneous Agents." *Games and Economic Behavior* 68, 703-715.

[12] Siegel, Ron. 2009. "All-Pay Contests." *Econometrica* 77, 71-92.

[13] Siegel, Ron. 2010. "Asymmetric Contests with Conditional Investments." *American Economic Review* 100, 2230-2260.

[14] Simon, Leo K. and Zame, William R. "Discontinuous Games and Endogenous Sharing Rules." *Econometrica*, July 1990, *58*(4), pp. 861-872.

[15] Szech, Nora. 2011 "Asymmetric All-Pay Auctions with Two Types." Mimeo.