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# "Competitive equilibrium as a bargaining solution: An axiomatic approach"

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## Abstract

The paper introduces an axiomatic characterization of a solution to bargaining problems. The bargaining problems addressed are specified by: (a) the preference relations of the bargaining parties (b) resources that are the subject of bargaining, and (c) a pre-specified disagreement bundle for each party that would result in case bargaining fails. The approach is ordinal, meaning that the solution is independent of the specific utility indices chosen to represent parties' preferences. We propose axioms that characterize a solution that matches each bargaining problem with an exchange economy whose parameters are derived from the problem, and assigns the set of equilibrium allocations corresponding to one equilibrium price vector of that economy. The axioms describe a solution that is the result of an impartial arbitration process, expressing the view that arbitration is a natural method to settle disputes in which agents have conflicting interests, but can all gain from a compromise.



Keywords: Bargaining, exchange economy, ordinal bargaining solution, competitive equilibrium.

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# 1 Introduction

In many business disputes the source of controversy between the conflicting parties concerns the distribution of assets among them. One of the most common methods for dispute resolution, alternative to litigation, is arbitration. When the conflicting parties choose to undergo arbitration they often reach quicker, more efficient, and sometimes less costly resolutions, compared to those they would have reached through conventional court proceedings. In contrast to litigation, which can be initiated whenever one of the conflicting parties decides to resort to court, a key feature of arbitration is that it can be initiated only upon the unanimous consent of the parties involved, and conducted according to principles they all agree on.

When a dispute over assets arises each party has a default allocation that can be achieved even when no agreement is reached. Surely a default of no assets can always be achieved, but in some cases, when a clear ownership of assets is involved, for instance when one of the disputing parties owns a real estate property, a more sophisticated disagreement result obtains. Nonetheless, in case of disagreement parties usually end up in a less favorable position, whereby the prolongation of the conflict delays the distribution of assets (or at least of those assets that remain after the parties attain their defaults). This could be the case, for instance, if the dispute ends up in a lengthy court litigation.

The current paper addresses disputes over multiple assets, involving several parties. Disputes are described by a bundle of assets that should be allocated to a number of parties, and by the preferences of those parties over assets. Preferences are only assumed to be ordinal, namely there is no notion of strength of preference and the only significant question when comparing two bundles is which of them is preferred over the other. It is supposed that each concerned party can prevent unilaterally an agreement (e.g. turn to court), in which case each of the involved parties will obtain a pre-defined default bundle. The problems handled are non-trivial in the sense that all parties maintain that the dispute may be resolved in more beneficial ways compared to their default results, hence all parties involved have an incentive to compromise. On the other hand, each party retains the option to influence the obtained compromise by imposing unilaterally the inferior default result. A solution is sought, that would assign to each dispute allocations of the assets under consideration.

Given that each entity involved in a dispute can cause the breakdown of an agreement, a solution would provide a viable method of dispute resolution only if parties are persuaded to accept it. Finding such a solution is our goal in the current essay, which offers an axiomatic characterization of dispute resolution.

When entering a business relationship, parties often sign a contractual agreement that requires them to resolve possible future disputes by means of an arbitration process. The contract furthermore specifies the principles by which arbitration will take place. Analogously to such a contract, our axioms formulate arbitration principles, describing a solution that results from an impartial and effective arbitration process. The axioms are then shown to be equivalent to a functional form of the solution, so that agreeing to an arbitration process that accords with the proposed principles is equivalent to employing the functional mechanism.

The problems we address consist of allocating goods to agents. The primary economic mechanism that is designed for that task is an exchange economy. Given initial bundles of goods, one bundle per each agent, a competitive equilibrium of the economy determines which allocation prevails: what goods each agent sells and how much each agent decides to consume. The solution concept characterized in this paper, termed *the market bargaining solution*, utilizes the market mechanism as an instrument. This is done by associating each dispute with an exchange economy, selecting an equilibrium price vector for that economy and assigning as a resolution all the equilibrium allocations that correspond to the selected equilibrium price vector (the solution therefore depends on the selection and is not unique). The exchange economy that is matched to a dispute is the one involving agents with the preferences of the disputing parties, each endowed with the corresponding party's default allocation plus an equal share of the remainder. Each concerned party is therefore considered to be in possession of his or her default result, with the remaining assets divided equally. In particular, if for every involved party the default result is receiving nothing (i.e. zero), the solution assigns equilibrium allocations (corresponding to the same equilibrium price vector) in an exchange economy where assets were originally distributed evenly. Our axiomatic characterization establishes that equilibrium allocations can be viewed as resolutions to a dispute rendered by an impartial arbitrator. An interesting feature of the development is that equilibrium prices are derived endogenously per dispute, and interpreted as representing an impartial arbitrator.

The problems handled in this paper are similar to classic bargaining problems as formulated by Nash [11]. Both types of problems concern the distribution of resources among a group of agents. In both cases a solution determines, for each problem, how resources are to be allocated to the parties involved, where each party is assumed to be able to reject compromise and unilaterally impose an inferior result on all parties involved. The axiomatic approach, aimed at attaining a consensus among disputing parties, is prevalent in the bargaining literature. The difference between the two types of problems lies in the assumptions made on the parties' preferences over resources, and in the formulation of feasible allocations and disagreement results.

Under the classic bargaining approach problems are formulated in terms of utilities. The default result, as well as feasible allocations of resources, are described in utility terms, suppressing any reference to the actual assets that underlie those utilities. Consequently, any two distinct economic environments that generate the same utility image will inevitably be assigned the same allocation as a solution. The utility-based approach relies on the assumption that meaningful information regarding the cardinality of parties' preferences is available, for instance due to preferences being defined over lotteries. However, for some problems such information is not attainable. Roemer, [15] and [16], discusses the implications of assuming that bargaining problems can be formulated in terms of utilities, and suggests a more general bargaining setup that is defined in terms of resources and preferences. This is the setup we employ in this paper, and accordingly, the solution we offer is specified in terms of assets. The difference between the resource-based bargaining approach and that of the classic bargaining literature can be illustrated through two simple examples.

Imagine two business partners engaged in a dispute over one acre of land, where in case of disagreement both get nothing. Each of the partners naturally wants to have as much of the land as possible. If no cardinal information regarding the partners' preferences is available, then their preferences for more land can be represented by any strictly increasing function of $x$, $x$ being the received fraction of the land. Consider two such representations, $u_1(x) = u_2(x) = x$ on one hand, and $u_1(x) = \sqrt{x}$ and $u_2(x) = 1-\sqrt{1-x}$ on the other, where $u_i$ stands for partner $i$'s utility from the land. Both representations induce the same utility-possibility set, namely the same set of utility allocations (these are the points in the triangle bounded within the origin and the points $(0, 1)$ and $(1, 0)$), and the same $(0, 0)$ default result in case of disagreement. Hence, every bargaining solution concept which employs the classic bargaining setup will match both problems with the same solution, that is with the same allocation of utilities. Symmetry of the utility-possibility set would lead to a fifty-fifty distribution of utilities under all well-known bargaining solution concepts. However, while under the first utilities representation this solution translates to assigning each of the partners half the acre of land, in the second utilities representation it entails allocating $1/4$ of the land to one partner and $3/4$ to the other. By contrast, a market bargaining solution will entail an equal division of the land regardless of the choice of specific utility representations, since underlying preferences over assets and the assets available for division remain unchanged.

Aside from assuming only ordinality of preferences, another difference between the two approaches is manifested in the conversion of disagreement results into bargaining powers. To demonstrate this difference, suppose two partners with identical Cobb-Douglas preferences disputing over one acre of land and one ton of wheat seed. Imagine that in case bargaining fails the first partner gets all the land while his rival gets nothing (let's say, because the first partner owned the land before entering the partnership). Any utility-based solution will ascribe equal bargaining powers to both partners since their default utilities are both zero. Yet it is unlikely that parties will concede to bargaining powers being interpreted like that, as this would mean ignoring differences related to investment size and initial ownership of assets. Our approach accounts for such differences by expressing the problem, including parties' default results, in assets terms.

An additional advantage of the market bargaining solution is that, in line with the Nash program, it offers ready-made implementation mechanisms, namely strategic-form games with Nash equilibria which constitute also competitive equilibria. Such mechanisms can be found, for instance, in Schmeidler [20] and Dubey [4].

On the negative side, an inevitable limitation of the suggested approach is that it applies only to bargaining problems whose corresponding exchange economies possess competitive equilibria. To that end, the paper characterizes a solution under assumptions that guarantee the existence of such an equilibrium. As is the case in market-related questions in general, here as well multiple equilibrium price vectors may exist. A market bargaining solution selects one of them, subject to some consistency requirements across different problems.

### 1.1 Literature review

Competitive equilibrium that originates from an equal division of resources is a prominent solution concept in the theory of fair allocations. It is known to satisfy many socially desirable attributes such as no envy, individual rationality, efficiency, and various consistency conditions, and is characterized axiomatically in Thomson, [23] and [24], Nagahisa [10], Nagahisa and Suh [9] and others. These axiomatizations, which naturally treat all parties equally, can be 'plugged in' as a solution to the kind of disputes over assets that we consider here, as long as the disagreement point is symmetric. The case of an asymmetric disagreement point, however, involving parties with disparate bargaining power, is not examined in those papers. Hence, they do not offer an axiomatic characterization of disputes over assets that we are interested in.

Furthermore, the fair allocations literature seeks to identify methods for resource distribution that are desirable from a social perspective. Axioms within that literature are formulated so that they look normatively appealing in the eyes of social planners, who commonly seek for solutions that focus primarily on traits such as fairness and stability. The problems addressed in this paper, however, require the cooperation of opposing agents rather than the consent of a social planner. To achieve agents' cooperation, we postulate axioms that describe individually appealing attributes of a solution, designed to gain the consent of the agents involved. For example, the fair allocations literature employs stability requirements which guarantee that sub-groups cannot do better on their own. But requirements of that sort are not appealing for agents that are concerned only with their own allocated assets. An agent will be indifferent to any re-distribution of assets among a sub-group that does not contain him or her, while preferring a profitable re-distribution of assets among a sub-group to which the agent belongs. Our characterization emphasizes reasoning pertaining to individualistic motives rather than to stability attributes.

Another well-known solution in the theory of distributive justice is the Pareto-efficient egalitarian-equivalent (PEEE) allocations, due to Pazner and Schmeidler [13]. These are Pareto efficient allocations that for each agent are equivalent preference-wise to the same fixed bundle. Thus, they are preference-wise equivalent to egalitarian allocations in some hypothetical economy. Pazner and Schmeidler suggest PEEE allocations as a solution for symmetric bargaining problems that are based on ordinal preferences, as in the setup employed here. Asymmetric problems are not addressed by their solution.

The PEEE solution, as well as the one we offer (for symmetric problems), consider as a starting point the equal division of the bundle in question, but from that point on the two solutions take different directions. Pazner and Schmeidler aim at obtaining an ex-post egalitarian solution, that is, their solution is designed to yield equitable results. By comparison, our solution concept is designed to yield individually optimal results. It is ex-ante egalitarian as resources are evenly divided among parties, thus guaranteeing equal opportunity to all of them. However, starting from this equal opportunity point onwards, the parties proceed to improve their welfare in a self-motivated manner. Such a procedure is more appropriate to the target audience of a bargaining solution, which constitutes of the selfishly-motivated parties rather than social planners. Continuing the incentive-compatible rationale, any allocation assigned by a market bargaining solution (for symmetric problems) is envy-free, namely every agent weakly prefers his or her allocated bundle over those of the other agents. PEEE allocations, on the other hand, may not be envy-free (when there are more than two parties to a dispute), and may even generate allocations in which one agent's allocated bundle dominates, good-by-good, another agent's bundle. In the course of a dispute, such a finding may lead to a denial of compromise.

This paper handles bargaining problems that are formulated in terms of assets and ordinal preferences, rather than in terms of utility values. Roemer, in [15] and [16], argues for the need of this kind of formulation, which allows for a richer variety of solution concepts. Roemer characterizes classic bargaining solution in this richer setup and demonstrates the strength of assumptions required for that characterization. Sertel and Yildiz [21] show that within the classic bargaining setup it is impossible to define a solution which for every utility image of an exchange economy assigns utilities of respective Walrasian allocations. Chen and Maskin [3] extend the economic setup to include production and characterize an egalitarian solution, equating agents' utilities, under a requirement that agents' utilities do not decrease following an improvement in technology. Perez-Castrillo and Wettstein [14] characterize a solution concept for economic environments that is formulated through endowments and ordinal preferences, generalizing attributes of the Shapley value. Their solution generalizes the egalitarian-equivalence concept of Pazner and Schmeidler to the case of non-equal initial positions, coinciding with the PEEE solution for problems with equal a-priori positions. The condition they use to characterize this solution involves contributions of agents to coalitions, namely to sub-groups of disputing agents. Nicolo and Perea [12] address two-persons bargaining problems, also formulated by means of resources and preferences (under some conditions their development can be applied to problems with any number of agents). The solution they define is also close in spirit to the egalitarian-equivalence concept of Pazner and Schmeidler [13]. It is axiomatized based on a form of monotonicity with respect to enlargement of possibility sets, and on a condition that binds together problems with different preferences.

In the problems that we have in mind agents agree in advance, prior to engaging in a joint business and before any dispute arises, to employ an arbitration mechanism as depicted in the axioms. It follows that the parties to be involved in a dispute are fixed. Thus, assumptions that involve possible changes concerning the nature of the involved agents (e.g. assumptions referring to sub-groups or to transformed preferences) will be deemed less relevant by agents.

Among the papers that take a preference-based approach to bargaining is Rubinstein, Safra and Thomson [18], that describes two-persons bargaining problems in terms of preferences over lotteries. In that paper a solution is defined by means of preferences, and is shown to coincide with the famous Nash bargaining solution [11] whenever preferences admit an expected utility representation. The result of Rubinstein, Safra and Thomson is extended in Grant and Kajii [5] to additional preference types. Essentially, the preferences and solutions considered in these works are cardinal. That is to say, preferences are represented by a utility function that is invariant only under positive linear transformations (where this type of uniqueness is the result of using a setup that contains lotteries), hence the solution to a bargaining problem may change following different order-preserving representations of preferences over assets. This contrasts with our framework which addresses preferences over assets which are ordinal in nature.

Our ordinal approach should not be confused with ordinal solutions to utility-formulated bargaining problems. The latter approach, as in Shapley [22] and Safra and Samet [19], among others, still considers classic bargaining problems that are described by means of agents' utility levels. Therefore, it cannot distinguish between different disputes that induce the same utilities image (as in the examples above). Another related paper, employing an economy as a solution device, is Trockel [25]. Trockel considers bargaining problems described in terms of utilities, and matches each one with an Arrow-Debreu economy the (unique) competitive equilibrium of which identifies with a corresponding asymmetric Nash bargaining solution.

Finally we mention a few well-known works in Bargaining Theory. Nash [11] was the first to formulate what is now known as the Nash Bargaining Problem: a two-persons setup in which parties can collaborate in a way that will be beneficial for both, and need to agree on the utility values that each will gain from this collaboration, otherwise they will obtain inferior disagreement values. Nash [11] phrases the problem in terms of utilities and offers an axiomatic treatment under the assumption that utility is cardinal (unique up to positive linear transformations). The axioms are shown to lead to a unique solution in terms of utilities. Later works suggest alternative axiomatizations and solutions to the Nash Bargaining Problem, still considering a utility-based formulation. Among those are Kalai and Somorodinsky [7], Kalai [6], and many others.

## 2 Symmetric bargaining problems

We first present the setup and notation, to be employed throughout the paper. Suppose a set of divisible goods, $\{1, \ldots, L\}$, and consider bundles of these goods which are elements in $\mathbb{R}^L_+$. For two bundles $x, y \in \mathbb{R}^L_+$ we write $x \geq y$ whenever $x_\ell \geq y_\ell$ for every $\ell = 1, \ldots, L$, and $x >> y$ whenever $x_\ell > y_\ell$ for every $\ell = 1, \ldots, L$. The set of allocations of bundles to a group of $n$ agents is $\mathcal{A}_n = (\mathbb{R}^L_+)^n$. An element of $\mathcal{A}_n$, termed *an allocation*, is denoted by $a = (a(1), \ldots, a(n))$, where $a(i) = (a_1(i), \ldots, a_L(i))$ is the bundle allocated to the $i$-th agent, containing quantity $a_\ell(i)$ of good $\ell$. For given resources $x = (x_1, \ldots, x_L)$, $\mathcal{A}_n(x)$ denotes the subset of $\mathcal{A}_n$ consisting of distributions of $x$ to $n$ parties, namely of allocations $a$ that satisfy, $\sum_{i=1}^{n} a_\ell(i) = x_\ell$ for every good $\ell$.

The bargaining problems addressed first are pairs $(x, (\succsim^i)_{i=1}^n)$, consisting of resources $x$ that need to be split between agents $i = 1, \ldots, n$, endowed with preferences over bundles given by binary relations over $\mathbb{R}^L_+$, $(\succsim^i)_{i=1}^n$. The asymmetric and symmetric parts of each $\succsim^i$ are respectively denoted by $\succ^i$ and $\sim^i$. It is first assumed that the disagreement point is that of the zero bundle for every agent. This case is generalized in Section 3 with reference to problems that contain a general disagreement point $d$, so that agent $i$ receives $d(i)$ in case bargaining fails.

As explained in the Introduction, in order for the market bargaining solution to be well defined it is essential that an exchange equilibrium exists in any problem considered. To guarantee this, a structural assumption is employed.

**A0. Structural assumption.**

Any bargaining problem $(x, (\succsim^i)_{i=1}^n)$ satisfies the following assumptions:

(1) The resources $x$ contain a positive quantity of each of the goods, namely $x >> 0$.[1]

(2) For every $i$ and $\succsim^i$ over $\mathbb{R}_+^L$ it holds that:

(a) $\succsim^i$ is complete and transitive.

(b) $\succsim^i$ is monotonic: for any $y, z \in \mathbb{R}_+^L$, if $y >> z$ then $y \succ^i z$.

(c) $\succsim^i$ is continuous: for every $y \in \mathbb{R}_+^L$ the sets $\{z \in \mathbb{R}_+^L \mid z \succsim^i y\}$ and $\{z \in \mathbb{R}_+^L \mid y \succsim^i z\}$ are closed.

(d) $\succsim^i$ is convex: for every $y, z \in \mathbb{R}_+^L$, if $y \succ^i z$ then $\lambda y + (1-\lambda)z \succ^i z$ for every $\lambda \in (0,1)$.

(e) $\succsim^i$ induces differentiable indifference curves: for $y \in \mathbb{R}_+^L$, $y >> 0$, define $D(y) = \{d \in \mathbb{R}^L \mid \exists \varepsilon > 0 \text{ s.t. } y + \varepsilon d \in \mathbb{R}_+^L \text{ and } y + \varepsilon d \succ^i y\}$. Then there is $v_y \in \mathbb{R}^L$ such that $D(y) = \{d \in \mathbb{R}^L \mid d \cdot v_y > 0\}$ (this definition is due to Rubinstein, [17]).

The conditions imposed on agents' preferences described in (2)(a) through (2)(d) are the standard Arrow-Debreu [1] conditions. The less standard part of the assumption, (2)(e), is in fact not required for the existence of an equilibrium but rather for the necessity part of our theorem (where the axioms are shown to be implied by the solution).

A *bargaining solution* is a correspondence $\varphi$ that assigns to every bargaining problem that satisfies the structural assumption above, a nonempty set of distributions of $x$ to the agents $i = 1, \ldots, n$. That is, $\varphi(x, (\succsim^i)_{i=1}^n)$ is a subset of $\mathcal{A}_n(x)$. Four attributes are assumed on a bargaining solution $\varphi$. The first assumption states that any allocation assigned by $\varphi$ is Pareto optimal. Any solution that does not comply with the Pareto condition can obviously be concertedly improved by the parties involved, therefore Pareto optimality constitutes a fundamental aspect in the plausibility of the solution.

**A1. Pareto.**

Let $(x, (\succsim^i)_{i=1}^n)$ be a bargaining problem. For every allocation $a \in \varphi(x, (\succsim^i)_{i=1}^n)$ and any allocation $a' \in \mathcal{A}_n(x)$, if $a'(i) \succsim^i a(i)$ for all parties $i = 1, \ldots, n$, then $a'(i) \sim^i a(i)$ for all

[1]If there is zero quantity of a good the problem can be reduced to one that deals with the allocation of a bundle not including that good.

$i = 1, \ldots, n$.

Agents in our framework are fully described by their preferences, and each agent is supposed to be concerned only with his or her own wellbeing. Thus, whenever two allocations yield for an agent the same utility level, the agent will regard them as equivalent. The second assumption maintains that all the agents deem all the allocations assigned by a given solution as equivalent. The solution therefore satisfies the property of *single-valuedness*, in the terminology of Moulin and Thomson [8]. Namely, the solution is, preference-wise, a singleton. If this were not the case, agents could not be expected to agree on any particular allocation assigned by a solution, since each allocation assigned could yield them a different utility level. The axiom further states that if two allocations are equivalent in any relevant respect (i.e., considered equivalent by any of the agents), either both of them are assigned by the solution or none of them is. A solution is therefore *a full correspondence*, as defined in Roemer [16].

**A2. Equivalence Principle.**
Suppose a bargaining problem $(x, (\succsim^i)_{i=1}^n)$ and two allocations, $a, a' \in \mathcal{A}_n(x)$. If $a, a' \in \varphi(x, (\succsim^i)_{i=1}^n)$ then $a(i) \sim^i a'(i)$ for every $i$. In the other direction, if, for every $i$, $a(i) \sim^i a'(i)$, then $a \in \varphi(x, (\succsim^i)_{i=1}^n)$ if and only if $a' \in \varphi(x, (\succsim^i)_{i=1}^n)$ .

We turn now to discuss the next axiom, which depicts the solution as the result of an impartial arbitration process.

A bargaining situation demands discretional agreement between all the agents, otherwise the disagreement point, unfavorable to all agents, will prevail. The need for unanimous consent among agents creates tension between their incentive to compromise and the power held by each agent to threaten the others with the breakdown of negotiations. An involvement of a third party is often required, and welcomed by agents, as a way to resolve this tension. A prevalent dispute resolution method, administered by a third party, is arbitration. In many business partnerships parties agree in advance that any future dispute among them will be resolved by arbitration, and specify beforehand its fundamentals.

Our third axiom requires that any allocation assigned by a solution be the result of arbitration conducted according to three basic principles. Hence, agents subscribing to this axiom accept this type of arbitration as a dispute resolution method. In conformity with the representation of parties to a dispute by their preferences, an arbitrator in our framework is modelled by a preference relation, satisfying (as do preferences of disputing parties) assumption A0(2). Any ranking of alternatives by such an arbitrating preference relation should be understood as reflecting the principles that guide the decisions of an arbitrator rather than an expression of this arbitrator's personal preferences.

An arbitrator will be interested in the allocation of resources among parties and not in single bundles per se. Accordingly, the requirements involving an arbitrating preference pertain to allocations as a whole. The requirements specify guidelines for the arbitral decision-making. Three standards are conveyed in our third axiom: (1) Arbitration should be unprejudiced, in the sense that no allocations should be eliminated a-priori; (2) Arbitration should treat all disputing parties fairly; (3) An arbitrator should be able to implement the allocations assigned as a resolution, making the solution feasible.

The three definitions below correspond to these three standards of arbitration. The axiom to follow applies these definitions to a preference relation that serves as an arbitrator. The first definition identifies a preference relation as unprejudiced whenever no allocation of resources to a number of parties is dominated by another allocation of the same resources to the same number of parties.

**Definition 1.** A binary relation $\succsim^*$ is *unprejudiced* if for every $x \in \mathbb{R}_+^L$ and $n \in \mathbb{N}$, and every $a, b \in \mathcal{A}_n(x)$, it cannot be the case that $a(i) \succ^* b(i)$ for every $i = 1, \ldots, n$.

When a preference relation reflects the decisions of an arbitrator, having one allocation dominated by another entails that the dominated allocation will never be assigned by the arbitrator. A dominated allocation will therefore be censored by the arbitrator before even considering a bargaining problem. Allocations, however, should be eliminated or chosen only upon observing the preferences of the parties involved in a dispute (for instance Pareto dominated allocations will be eliminated after observing parties' preferences), otherwise solutions may be renounced prior to approaching a dispute, even if they may turn out to be favorable to the parties involved.

The second definition states that a preference relation perceives an allocation as fair whenever the relation is indifferent between all the bundles comprising the allocation.

**Definition 2.** A binary relation $\succsim^*$ *perceives an allocation* $a \in \mathcal{A}_n$ *as fair* if for any $i$ and $j$, $a(i) \sim^* a(j)$.

Definition 2 allows us to express the most basic standard of arbitration, namely an impartial treatment of parties. In symmetric problems, when agents' claims are equal, agents will expect an impartial arbitrator to rule fairly and assign allocations which the arbitrator deems equitable. Stated differently, agents will not respect the authority of an arbitrator who intends to assign better bundles to peers who a-priori should be treated equally.[2] Here it should be

[2]This will no longer be the case for asymmetric problems, where different agents have different default

noted that while the fairness of an arbitration process is judged from the point of view of an arbitrating preference, our result will establish that agents themselves find their share under our solution at least as desirable as their opponents' shares (an attribute known as *no envy*).

In order for an arbitration process to be effective the arbitration decision should be binding. Each involved party must commit to accept the arbitration decision, otherwise the arbitration agreement may eventually not ensue, and the less favorable default result will prevail. The third definition phrases conditions under which an allocation can be enforced by a preference relation. The definition refers to bargaining problems derived from an original problem through replacing all agents but one by the same preference relation. A preference relation is said to be able to implement an allocation in a bargaining problem, through a solution $\varphi$, if the allocation is contained in the solution of each such replaced problem. This definition is employed in the following axiom so as to maintain that an arbitrator can implement an allocation as a dispute resolution by negotiating the allocation with each of the parties separately.

**Definition 3.** A binary relation $\succsim^*$ *can implement an allocation* $a \in A_n(x)$ *in a problem* $(x, (\succsim^i)_{i=1}^n)$*, through a solution* $\varphi$*,* if for any agent $i$, $a \in \varphi(x, (\succsim^i, \underbrace{\succsim^*, \ldots, \succsim^*}_{n-1 \text{ times}}))$ (where $a(i)$ is still allocated to agent $i$).

The next axiom, Impartiality, describes a solution as the result of an impartial and binding arbitration process, by stating that any allocation in the solution to a dispute admits an arbitrating preference that fulfills all three standards specified above. Following the equivalence principle, stating a condition that applies to *some* allocation in a solution implies that the condition applies to *the* unique preference-levels assigned by this solution. Hence, Impartiality implies that the (preference-wise) solution is a resolution to the dispute assigned by an unprejudiced and fair arbitrator that has the power to enforce it. When disputing parties adopt this axiom they commit to resolving disputes by means of an arbitration conducted in line with the three basic principles listed above. The parties agree on an arbitrator chosen from among candidates who approach the problem open-mindedly.[3] They expect that the arbitrator be just and treat all parties equitably. Finally, by subscribing to the axiom, each agent acknowledges that the same solution will befall in case he or she bargains privately with the arbitrator. Mutual commitment of all agents to this condition guarantees that the solution to a dispute can be implemented by the arbitrator through separate negotiations with the parties involved.

allocations. For asymmetric problems this definition will be adjusted to accommodate different starting points for different agents.

[3]In real-life disputes, arbitrators are frequently chosen from among members of a specialized institution, such as the American Arbitration Association (AAA), so that parties can trust their impartiality. As stated on the AAA website, "These neutrals are bound by AAA established standards of behavior and ethics to be fair and unbiased".

**A3. Impartiality.**
Let $(x, (\succsim^i)_{i=1}^n)$ be a bargaining problem. Any allocation $a \in \varphi(x, (\succsim^i)_{i=1}^n)$ admits an unprejudiced binary relation, that conforms to A0(2), perceives $a$ as fair, and can implement it in the problem $(x, (\succsim^i)_{i=1}^n)$, through $\varphi$.

Note that we do not require that an arbitrator corresponding to an allocation, in the manner described in the axiom, be unique. Nevertheless, our theorems will ultimately imply such uniqueness. Although there may be multiple adequate arbitrators per problem, the characterized solution selects one arbitrator, who is the only one corresponding to the allocations assigned as a solution to that problem. The theorems will further establish that the selected arbitrator is able to negotiate a solution with any sub-group of parties, namely the same solution will result in case the arbitrator replaces any number of agents.

The last axiom is a consistency assumption referring to a certain form of re-scaling. It compares between solutions to two problems, an original problem and its replication. By the latter we mean the problem in which both the number of agents and the resources to be allocated are replicated by the same factor, so that the replicated problem consists of allocating $k$ times the original resources to a group of agents comprised of $k$ copies of each original agent, as formally defined below.

**Definition 4.** For $k \in \mathbb{N}$, the *k-replication* of a bargaining problem $(x, (\succsim^i)_{i=1}^n)$ is the bargaining problem $(kx, \underbrace{((\succsim^i)_{i=1}^n, \ldots, (\succsim^i)_{i=1}^n}_{k \text{ times}}))$.

By having multiple representatives of each original preference we create a situation whereby each agent's claims are replicated. The resources in dispute are replicated as well in the same manner. The combined effect of both is akin to that of re-scaling. Replication Invariance asserts that a solution is invariant under this sort of re-scaling. First, the re-scaled solution is contained in the solution to the re-scaled problem, hence, by the Equivalence Principle (**A2**), the solutions to the original and to the re-scaled problems are preference-wise equivalent. Second, if an arbitrator can serve in the original problem then he or she can also serve in its re-scaling. This condition is similar to an attribute by the same name that was introduced by Thomson (see the survey [24] and the references therein).

**A4. Replication Invariance.**
Let $(x, (\succsim^i)_{i=1}^n)$ be a bargaining problem and $a \in \varphi(x, (\succsim^i)_{i=1}^n)$ an allocation in its solution. Then:

(a) The $k$-replicated allocation $a^k = (\underbrace{a, \ldots, a}_{k \text{ times}})$ is in the solution to the $k$-replication of $(x, (\succsim^i)_{i=1}^n)$.

(b) If an unprejudiced binary relation, conforming to A0(2), perceives $a$ as fair and can implement it in the problem $(x, (\succsim^i)_{i=1}^n)$, through $\varphi$, then it can also implement $a^k$ in the $k$-replicated problem, through $\varphi$.

Our first result shows that under the structural assumption (**A0**), whenever the default result is nothing for all (i.e. in the case of a zero disagreement point, and in fact with any symmetric disagreement point), axioms **A1-A4** characterize a market bargaining solution. Among these axioms, Pareto (**A1**) is a basic condition without which a solution is simply senseless, and the bargaining parties will never accept its assigned allocations. Replication Invariance (**A4**) and the Equivalence Principle (**A2**) guarantee the consistency of a solution from two aspects. The first aspect is consistency with respect to re-scaling, and the second is consistency with respect to the representation of a problem, whereby a solution depends entirely on preferences. All these three axioms are standard in the literature. The primary assumption in the characterization of a solution is Impartiality (**A3**), which describes a solution as the result of an impartial arbitration process. This is where our characterization (of symmetric problems) departs from existing ones in the literature. Our theorem states that parties' consent to resolve disputes through arbitration that conforms to the axioms is the same as endowing each party with an equal share of resources and allowing parties to trade among themselves. Each party will obtain its share under a competitive equilibrium allocation that corresponds to one selected equilibrium price vector. As a result, a dispute resolution under the axioms satisfies the appealing 'no envy' attribute: agents prefer their own share over others' shares.

Note that the dispute resolution that we offer is not unique, as multiple equilibrium price vectors may exist per dispute. The axiomatization does not in general mandate that any specific equilibrium price vector of the induced exchange economy be selected. It only dictates that the selection is consistent in the two following manners.

**Definition 5.** A bargaining solution $\varphi$ is *replication consistent* if the equilibrium allocations it assigns, given any bargaining problem and its $k$-replications ($k \in \mathbb{N}$), correspond to the same vector of equilibrium prices.

**Definition 6.** A bargaining solution $\varphi$ is *implementation consistent* if whenever it assigns allocations that correspond to a vector $p$ of equilibrium prices, given a bargaining problem, then the allocations it assigns to the replaced problem that is created when all agents but one are replaced with a $p$-linear preference[4] correspond to the same vector of equilibrium prices $p$.

[4]That is, with the preference that admits a representation $u(y) = p \cdot y$ for every $y \in \mathbb{R}_+^L$.

Replication consistency is important in order to avoid cases where essentially equivalent descriptions of the problem result in different equilibrium prices and thereby different utility levels for the involved parties. If this were the case, the mere re-scaling of the problem, through replicating the bargaining powers of parties on the one hand and correspondingly replicating the available quantities of resources on the other hand, would yield different utility levels for different agents. Such inconsistencies could open the door to manipulations in the description of problems. Implementation consistency is implied by our axiomatic characterization. It limits the choice of equilibria only in some problems where all agents but one are linear with the same characterizing vector of coefficients.

The following theorem states that any solution that satisfies the axioms matches each dispute with an exchange economy and selects one equilibrium price vector in that economy. The solution then assigns as a resolution of the dispute all the equilibrium allocations that are distributions of $x$, corresponding to the selected price vector (i.e., in case of a good with a zero price the solution will assign a subset of the corresponding equilibrium allocations, composed of those equilibrium allocations that do not generate excess supply). Following Arrow and Debreu [1] our structural assumption maintains that such a solution indeed exists.

**Theorem 1.** *Let $\varphi$ be a bargaining solution to problems that satisfy the structural assumption* ***A0****. The following two statements are equivalent:*

*(i) $\varphi$ satisfies assumptions* ***A1-A4****.*

*(ii) For any bargaining problem $(x, (\succsim^i)_{i=1}^n)$, $\varphi$ assigns all the equilibrium allocations that distribute $x$, corresponding to one equilibrium price vector in a pure exchange economy where agents hold preferences $(\succsim^i)_{i=1}^n$ and each of them is endowed with $x/n$. In addition, $\varphi$ is replication-consistent and implementation-consistent.*

The theorem implies that an impartial arbitrator is selected for every bargaining problem. This is a linear preference relation whose vector of coefficients is an equilibrium price vector in the corresponding exchange economy. We illustrate the market solution to bargaining problems (both for the symmetric and the asymmetric cases) in Subsection 3.1.

## 3 Asymmetric bargaining problems

Thus far the characterization concerned only bargaining problems with an all-zeros disagreement point: in case parties refuse to compromise and negotiations break down, everyone ends up with nothing. But bargaining situations can involve default options other than zero. Disagreement in such situations would leave each party with some pre-specified bundle, signifying the different bargaining powers of parties. It turns out that our approach can accommodate these cases by altering only the definition of a fair allocation as perceived by a preference relation. With a general disagreement result agents start off with different fallback options, some perhaps more favorable than others. For instance, one agent may be in possession of some of the assets while others are not. In such cases agents will expect previous ownerships of assets to be taken into account in a solution, perceiving settlements of the dispute which ignore different a-priori rights as unfair. The notion of fairness is hence amended to acknowledge different starting positions of agents implied by unequal default allocations.

In order to address problems with a general disagreement point the previous framework is extended. A bargaining problem is now a triplet, $(x, (\succsim^i)_{i=1}^n, d)$, consisting of resources $x$ that need to be split between parties with preferences over bundles, $(\succsim^i)_{i=1}^n$, and of a disagreement point $d = (d(1), \ldots, d(n))$, $d(i)$ being the bundle to be allocated to party $i$ in case bargaining fails. An addition to the structural assumption **A0** is required, asserting that for every good $\ell$ the disagreement point does not exhaust the available quantity of this good. Combined with monotonicity (part (b) of **A0**) it implies that problems are non-trivial, in the sense that some allocations are deemed by everybody as better than the default. Technically speaking, this additional assumption guarantees that an equilibrium exists for the economy that corresponds to any extended problem.

**A0d. Structural assumption.**

Any bargaining problem $(x, (\succsim^i)_{i=1}^n, d)$ satisfies **A0**, as well as $x >> \sum_{i=1}^{n} d(i)$.

A bargaining solution $\varphi$ is now a correspondence that assigns to every bargaining problem $(x, (\succsim^i)_{i=1}^n, d)$ that satisfies **A0d** a set of allocations in $\mathcal{A}_n(x)$.

The essence of different disagreement bundles is that agents possess different bargaining powers. Our axioms characterize a solution that takes different bargaining powers into account by translating them into endowments in the respective exchange economy. The endowment of an agent in that economy becomes his or her disagreement bundle plus an equal share of the remainder. These endowments are akin in spirit to the 'Contested Garment Principle' of Aumann and Maschler [2], prescribing equal division of the contested amount, if each party $i$ is taken to claim $d_i + (x - \sum_{j=1}^n d_j)$ of the resources under consideration. Such claims reflect the supposition that disputing parties accept their peers' rights over their default bundles and thus claim only what remains of the resources after deducing others' disagreement bundles.

The Pareto and the Equivalence Principle assumptions remain unchanged (only the problems they refer to are amended to contain a disagreement point $d$). As for Impartiality, the notion of an allocation being perceived as fair by a preference relation is generalized to identify the fairness of an allocation *given a disagreement point*. The definition no longer requires that the relation finds all agents' bundles to be equivalent, as that would eliminate any reference to differing bargaining powers, but rather it requires that the *surplus* generated for each agent be judged as equivalent by the relation. Thus, an allocation deemed as fair by a preference relation, given an asymmetric default result, distributes resources in a fashion perceived as inequitable by that preference relation, but only as a result of the agents' rights being a-priori different.

**Definition 7.** A binary relation $\succsim^*$ *perceives an allocation* $a \in \mathcal{A}_n$ *as fair given a disagreement point* $d$ if there exists a bundle $t \in \mathbb{R}^L_+$ such that for all $i$, $a(i) \sim^* d(i) + t$.

The assertion that a preference relation can implement an allocation in a problem through a solution remains essentially the same, requiring that the allocation still be assigned by the solution when the preference relation negotiates the problem separately with any of the agents. The difference is that the replacing relation now inherits the replaced relations' disagreement bundles as well.

**Definition 8.** A binary relation $\succsim^*$ *can implement an allocation* $a \in A_n(x)$ *in a problem* $(x, (\succsim^i)_{i=1}^n, d)$*, through a solution* $\varphi$*,* if for any agent $i$, $a \in \varphi(x, (\succsim^i, \underbrace{\succsim^*, \ldots, \succsim^*}_{n-1 \text{ times}}), d)$ (where $d(i)$ is still the default result of agent $i$, $a(i)$ is still this agent's allocated bundle under $a$, and for each $j \neq i$, the copy of $\succsim^*$ whose default bundle is $d(j)$ receives $a(j)$).

The adjusted Impartiality axiom still maintains that a solution is an implementable outcome of an unprejudiced arbitration process. Now, however, it is required that parties be treated fairly after taking into consideration their different initial positions.

**A3d. Impartiality.**
Let $(x, (\succsim^i)_{i=1}^n, d)$ be a bargaining problem. Any allocation $a \in \varphi(x, (\succsim^i)_{i=1}^n, d)$ admits an unprejudiced binary relation, that conforms to A0(2), perceives $a$ as fair given $d$, and can implement it in the problem $(x, (\succsim^i)_{i=1}^n, d)$, through $\varphi$.

The last axiom, Replication Invariance, remains essentially the same only with $k$-replicated problems that take into account the disagreement point, as formulated below.

**Definition 9.** For $k \in \mathbb{N}$, the *k-replication* of a bargaining problem $(x, (\succsim^i)_{i=1}^n, d)$ is the bargaining problem $(kx, (\underbrace{(\succsim^i)_{i=1}^n, \ldots, (\succsim^i)_{i=1}^n}_{k \text{ times}}), (\underbrace{d, \ldots, d}_{k \text{ times}}))$.

**A4d. Replication Invariance.**
Let $(x, (\succsim^i)_{i=1}^n, d)$ be a bargaining problem and $a \in \varphi(x, (\succsim^i)_{i=1}^n, d)$ an allocation in its solution. Then:

(a) The $k$-replicated allocation $a^k = (\underbrace{a, \ldots, a}_{k \text{ times}})$ is in the solution to the $k$-replication of $(x, (\succsim^i)_{i=1}^n, d)$.

(b) If an unprejudiced binary relation, conforming to A0(2), perceives $a$ as fair given $d$ and can implement it in the problem $(x, (\succsim^i)_{i=1}^n, d)$, through $\varphi$, it can also implement $a^k$ in the $k$-replicated problem, through $\varphi$.

A characterization of a market bargaining solution for the general type problems ensues. In this more general case, under **A0d**, axioms **A1**,**A2**,**A3d** and **A4d** are equivalent to a market bargaining solution with differing bargaining powers. As in the previous theorem, here again a solution as depicted indeed exists, according to Arrow and Debreu [1]. The only fundamental difference compared to the symmetric case elaborated in the previous section lies in the identification of fair allocations, which are now equitable only once defaults are accounted for. The theorem implies that accepting the arbitration principles encapsulated in the above axioms is equivalent to allotting each agent his or her default allocation plus an equal share of the remainder, and letting the agents trade. In particular, the theorem implies that a market bargaining solution is an individually rational solution, since each agent prefers his or her assigned bundle over the default bundle.

**Theorem 2.** *Let $\varphi$ be a bargaining solution to problems $(x, (\succsim^i)_{i=1}^n, d)$ that satisfy the structural assumption **A0d**. The following two statements are equivalent:*

*(i) $\varphi$ satisfies assumptions **A1,A2,A3d** and **A4d**.*

*(ii) For any bargaining problem $(x, (\succsim^i)_{i=1}^n, d)$, $\varphi$ assigns the set of all equilibrium allocations that distribute $x$, corresponding to one equilibrium price vector in a pure exchange economy where agent $i$ holds preferences $\succsim^i$ and is endowed with $d(i) + \frac{1}{n}\left(x - \sum_{j=1}^{n} d(j)\right)$, $i = 1, \ldots, n$.*
*In addition, $\varphi$ is replication-consistent and implementation-consistent.*

As in the previous theorem here again each dispute admits an impartial arbitrator in the form of a linear preference relation, characterized by the selected vector of equilibrium prices. As stated in the Introduction, as far as we are aware of, this is the first axiomatic characterization of a competitive equilibrium from a nonequal division of resources.

Finally, we would like to conclude with an example.

### 3.1 Example

Imagine two business partners who need to decide on the distribution of one acre of land, one million dollars and two tons of wheat seed. The resources to be split are $(1, 1, 2)$, where the coordinates denote acres of land, millions of dollars and tons of wheat seed, respectively. The preference relations of the two partners involved are denoted by $\succsim^1$ and $\succsim^2$. For the sake of the illustration suppose that these preferences belong to the Cobb-Douglas class with parameters $(1/2, 1/3, 1/6)$ for $\succsim^1$ and parameters $(1/4, 1/2, 1/4)$ for $\succsim^2$.

First consider the situation in which if bargaining fails both partners receive nothing (the symmetric all-zeros case). The market bargaining solution for that case suggests that partners be allocated their equilibrium bundles in the exchange economy in which each is endowed with half the resources, namely with $(1/2, 1/2, 1)$. Given the partners' Cobb-Douglas preferences, the market bargaining solution allocates $(2/3, 2/5, 4/5)$ to the first partner and $(1/3, 3/5, 6/5)$ to the second. This will in fact be the solution to every problem that assigns the partners with identical bundles in case bargaining fails.

To understand how a solution depends on parties' bargaining power we compare the above situation to that in which if bargaining fails the first partner ends up with the bundle $(0.25, 0, 0.5)$ and the second with $(0.5, 0.5, 0)$. The market bargaining solution for that problem entails that the partners be allocated their equilibrium allocations in the exchange economy in which the first partner is endowed with $(0.375, 0.25, 1.25)$ and the second partner with $(0.625, 0.75, 0.75)$. Solving the equilibrium for these parameters implies that according to the market bargaining solution the first partner will be allocated the bundle $(6/11, 2/7, 4/7)$ and the second will be allocated $(5/11, 5/7, 10/7)$. As can be seen, the first partner's allocation under the given asymmetric circumstances is worse than his or her allocation in the symmetric case. The reason is that the first partner entered the bargaining situation with less bargaining power than the second partner, thus in an inferior position to his or her own position in the symmetric bargaining situation.

The difference between the partners' allocations in the symmetric versus the asymmetric situations stands in sharp contrast to the results obtained under classic bargaining solutions. To further illustrate the difference, suppose that in the above example the two partners hold identical Cobb-Douglas preferences, characterized by parameters $(1/2, 1/3, 1/6)$. When disagreement bundles are identical (as in the all-zeros disagreement point, for instance) both partners possess the same bargaining power and as in all classic bargaining solutions the market bargaining solution will yield them equal bundles. Now, however, assume that the bargaining situation is such that in case bargaining fails the first partner gets all the land and all the wheat seeds while the second one gets nothing.[5] Any utility-based solution cannot distinguish between those two problems since both partners' default utilities, in both bargaining stories (i.e. the symmetric and the seemingly asymmetric one), are zero. Thus, any utility-based solution will still prescribe an even distribution of resources. The market bargaining solution, on the other hand, will assign as a resolution the equilibrium allocation in the exchange economy in which the first partner is endowed with $(1, 0.5, 2)$ and the second partner with $(0, 0.5, 0)$. The market resolution to this dispute will therefore consist of allocating one sixth of the resources to the second partner, and five times as much to the first partner.

# 4 Proofs

## 4.1 Proof of Theorem 1

### 4.1.1 Proving that (i) implies (ii)

Let $\varphi$ be a bargaining solution to problems that conform to the structural assumption **A0**, and suppose that $\varphi$ satisfies assumptions **A1-A4**.

**Step 1:** It is first shown that if a binary relation $\succsim^*$ conforms to A0(2) and is unprejudiced, then $\succsim^*$ must be a linear relation, satisfying monotonicity. That is to say, there is a vector of coefficients $m \in \mathbb{R}^L_+$ such that $\succsim^*$ ranks bundles $y \in \mathbb{R}^L_+$ according to the utility function $u_*(y) = m \cdot y$.

**Lemma 1.** *Suppose an unprejudiced binary relation $\succsim^*$, that conforms to A0(2). Then there is a vector $m \in \mathbb{R}^L_+$ such that $y \succsim^* z$ if and only if $m \cdot y \geq m \cdot z$.*

Proof. It is first proved that each indifference curve of $\succsim^*$ is a hyperplane. Let $t >> 0$ and consider a tangent hyperplane to the indifference curve of $\succsim^*$ that goes through $t$. Denote its characterizing coefficients by $m \in \mathbb{R}^L_+$ (a non-negative such tangent exists on account of A0(2b)). Note that given that $\succsim^*$ satisfies A0(2) (and thus satisfies in particular convexity) it holds that $t \succsim^* z$ for every $z \in \mathbb{R}^L_+$ for which $m \cdot z = m \cdot t$. Suppose on the contrary that the $\succsim^*$-indifference curve going through $t$ is not a hyperplane, so that there is $y \in \mathbb{R}^L_+$ such that both $y \sim^* t$ and $m \cdot y > m \cdot t$.

Let $\lambda$ be a number greater than 1. Since $m \cdot y > m \cdot t$, there is $\varepsilon(\lambda) > 0$ such that (i) $\varepsilon(\lambda)\lambda m \cdot t = (\lambda - 1)m \cdot (y - t)$, and (ii) $\varepsilon(\lambda) \to 0$ as $\lambda \to 1$. Set, $z = \lambda(1 + \varepsilon(\lambda))t + (1 - \lambda)y$,

[5] These disagreement bundles do not satisfy the condition in A0d, but the same essential point can be made with *almost* all the land, and *almost* all the wheat seeds given to the first partner.

which for $\lambda$ close enough to 1 is in $\mathbb{R}^L_+$. Thus, $m \cdot z = m \cdot t$, which implies $t \succsim^* z$.

Note that $(1+\varepsilon(\lambda))t$ is on the interval connecting $y$ and $z$. More precisely, $(1+\varepsilon(\lambda))t = (1-\frac{1}{\lambda})y + \frac{1}{\lambda}z$. Let $\lambda = \frac{n}{k}$ be a rational number, with $n > k > 0$ being integers. Letting $\varepsilon = \varepsilon(\frac{n}{k})$ we obtain,

$$(1+\varepsilon)t = (1-\frac{k}{n})y + \frac{k}{n}z \ . \tag{1}$$

When $\frac{k}{n}$ is sufficiently close to 1, $z >> 0$ (because $t >> 0$) and $y - \varepsilon t, z - \varepsilon t \geq 0$.

We now construct allocation $b \in \mathcal{A}_n(nt)$ by assigning $b(i) = z - \varepsilon t$ for $i = 1, ..., k$ and $b(i) = y - \varepsilon t$ for $i = k+1, ..., n$. Allocation $b$ is indeed in $\mathcal{A}_n(nt)$, because by Eq. (1), $k(z-\varepsilon t)+(n-k)(y-\varepsilon t) = nt$. We compare $b$ to the constant allocation $a \in \mathcal{A}_n(nt)$, assigning $a(i) = t$, for every $i = 1, ..., n$.

Recall that $t \sim^* y$ and $t \succsim^* z$. According to A0, $\succsim^*$ is monotonic and due to $t >> 0$, we obtain $a(i) \succ^* b(i)$ for every $i = 1, ..., n$. This is a contradiction to the assumption that $\succsim^*$ is unprejudiced. We conclude (after applying also the continuity of the relation) that $\succsim^*$ generates hyperplane indifference curves.

It remains to show that the hyperplane indifference curves are parallel. Suppose on the contrary two indifference curves which are not parallel. Let $t_1 >> 0$ lie on the lower curve, and let $t_2$ be the point on the second indifference curve which lies on the ray from the origin that crosses $t_1$, namely $t_2 = (1+\alpha)t_1$ for some $\alpha > 0$.

As the two hyperplane indifference curves in question are non-parallel, there is a vector $v$ such that both $t_2 + v$ and $t_2 - v$ are in the hyperplane of $t_2$, while $t_1 + v$ is above and $t_1 - v$ is below the hyperplane of $t_1$. In terms of preferences it implies that $z_2 := t_2 + v \sim^* t_2$, while $z_1 := t_1 - v \prec^* t_1$. Denote $t := (t_1 + t_2)/2$. Note that $2t = z_1 + z_2$, implying that $(z_1, z_2)$ is a split of $2t$ into two. Furthermore, $z_2 \sim^* t_2$ but $t_1 \succ^* z_1$. Due to continuity and monotonicity there is $\lambda > 0$ such that $w := z_1 + \lambda t \sim^* t_1$. We thus obtain, $w + z_2 = z_1 + \lambda t + z_2 = (2+\lambda)t$.

We now construct an allocation $b \in \mathcal{A}_2(2t)$: $b(1) = w - (\lambda/2)t$ and $b(2) = z_2 - (\lambda/2)t$. Note that $b(1)$ is strictly smaller than $w$ and $b(2)$ is strictly smaller than $z_2$, and therefore $w \succ^* b(1)$ and $z_2 \succ^* b(2)$. By comparing allocation $b$ to the allocation $a \in \mathcal{A}_2(2t)$, composed of $a(1) = t_1$ and $a(2) = t_2$, a contradiction is inflicted upon the assumption that $\succsim^*$ is unprejudiced. This leads us to assert that $\succsim^*$ is characterized by parallel hyperplane indifference curves, namely there is $m \in \mathbb{R}^L_+$ (where nonnegativity of the components of $m$ stems from monotonicity of the relation) such that for every two bundles $y$ and $z$, $y \succsim^* z$ if and only if $m \cdot y \geq m \cdot z$. ■

Consider a bargaining problem $(x, (\succsim^i)_{i=1}^n)$ and an allocation in its solution, $a \in \varphi(x, (\succsim^i)_{i=1}^n)$. According to Impartiality (**A3**) there exists an unprejudiced binary relation $\succsim^*$, that conforms to A0(2), can implement $a$ in this problem through the solution $\varphi$ and is indifferent between all the bundles allocated under $a$. By the above lemma this relation is linear. If we denote by $m = (m_1, \ldots, m_L)$ the vector of coefficients characterizing this relation, then $m \cdot a(i)$ is constant across all agents $i = 1, \ldots, n$, hence $m \cdot a(i) = \dfrac{m \cdot x}{n}$ for every $i$. Denote $b = \dfrac{m \cdot x}{n}$.

For an agent $i$, consider the problem $(x, (\succsim^i, \underbrace{\succsim^*, \ldots, \succsim^*}_{n-1 \text{ times}}))$, created when each agent $j \neq i$ is replaced by the relation $\succsim^*$. According to Impartiality (**A3**) the allocation $a$ is contained in the solution to this replaced problem, where $a(i)$ is allocated to $i$ and $a(j), j \neq i$ is allocated to the $\succsim^*$-clones.

**Step 2:** Let $y \in \mathbb{R}_+^L$ be such that $x \geq y$ and $m \cdot y = b$. Consider the following distribution of $x$ to agent $i$ and to the $(n-1)$ clones of $\succsim^*$: $i$ is allocated $y$, and each $\succsim^*$ clone is allocated $\frac{1}{n-1}(x - y)$. Since $m \cdot \frac{x-y}{n-1} = b$, every agent with preferences $\succsim^*$ is indifferent between his or her share under $a$, and $\frac{x-y}{n-1}$. The Pareto assumption implies that $a(i) \succsim^i y$. We conclude that $a(i) \succsim^i y$ for every bundle $y$ satisfying $x \geq y$ and $m \cdot y = b$.

**Step 3:** Consider the doubled problem, $(2x, (\succsim^1, \succsim^1, \ldots, \succsim^n, \succsim^n))$, consisting of allocating twice the resources to a double-sized group of agents in which there are two clones of every preference $\succsim^i$, $i = 1, \ldots, n$. According to Replication Invariance (**A5**) the allocation giving $a(i)$ to every $i$-clone is included in the solution to the doubled problem, and can also be implemented by $\succsim^*$ in that problem, under the solution $\varphi$. Denote the doubled allocation by $a^d$. The conclusion of the previous step is now applied to the doubled problem (in which the resources to be split are $2x$). It results that for every agent $i$ and any bundle $y$ such that $2x \geq y$ and $m \cdot y = b$, $a(i) \succsim^i y$.

By iterating the same argument (with $4x, 8x$, and so on) it may be concluded that any agent $i$ prefers $a(i)$ over any bundle $y$ whose worth (w.r.t. $m$) is $b$. In other words, we obtain that for every bargaining problem $(x, (\succsim^i)_{i=1}^n)$ and every agent $i$, any allocation in $\varphi(x, (\succsim^i)_{i=1}^n)$ maximizes $i$'s utility among all the bundles whose worth w.r.t. $m$ is the same, for some such vector of coefficients $m = (m_1, \ldots, m_L)$. This is precisely the formulation of the constrained optimization problem in a pure exchange economy where all agents' endowments coincide and the competitive equilibrium price vector is $m$.

**Step 4:** We show here that a vector $m$ as above does exist. By **A0** (see Arrow and Debreu [1]) there exists a competitive equilibrium for the above exchange economy. Thus, a vector $m$ exists as desired – it is a prevailing competitive price system. The allocation $a$ is an equilibrium allocation corresponding to the equilibrium prices given by $(m_1, \ldots, m_L)$, under equal endowments for all agents. Following the Equivalence Principle (**A2**), all the equilibrium allocations that correspond to the prices $(m_1, \ldots, m_L)$ are included in the solution. The next claim shows that these are the only allocations included in the solution.

**Claim 1.** *Suppose that $a \in \varphi(x, (\succsim^i)_{i=1}^n)$ is an equilibrium allocation corresponding to an equilibrium price vector $m$ (for some endowments, either $x/n$ or asymmetric ones). Then for any $a' \in \varphi(x, (\succsim^i)_{i=1}^n)$, $a'$ is also an equilibrium allocation that corresponds to the same price vector $m$.*

Proof. Let $a = (a(i))_{i=1}^n$ and $a' = (a'(i))_{i=1}^n$ be two allocations in $\varphi(x, (\succsim^i)_{i=1}^n)$, and suppose that $a$ is an equilibrium allocation corresponding to the price vector $m$ in an exchange economy with some endowments (uniform or other). The Equivalence Principle (**A2**) entails that all the agents are indifferent between the two allocations: $a(i) \sim^i a'(i)$ for every $i$.

Since $a$ is an equilibrium allocation, $a(i)$ - the share of $i$ - optimizes $i$'s utility subject to $i$'s budget constraint. Suppose that there is an agent $j$ whose allocation $a'(j)$ satisfies $m \cdot a'(j) > m \cdot a(j)$. Following the fairness of allocations assigned by the solution, there must be another agent $i$ whose allocation $a'(i)$ satisfies $m \cdot a'(i) < m \cdot a(i)$. By **A0**, all preferences are monotonic. If $m \cdot a'(i) < m \cdot a(i)$, then $i$ could add a strictly positive bundle to $a'(i)$ without violating the budget constraint. This would improve $i$'s utility, meaning that $a(i)$ would not be optimal for $i$ under the budget constraint. Therefore, we must conclude that $m \cdot a'(i) = m \cdot a(i)$ for every $i$. Thus $a'(i)$ is optimal for every $i$ under the budget constraint corresponding to $m$, which makes the entire allocation $a'$ an equilibrium allocation under the price vector $m$. ■

We conclude that, given a bargaining problem, a solution $\varphi$ to this problem equals the set of all the equilibrium allocations corresponding to some vector of equilibrium prices $m$.

**Step 5:** The last step in the proof consists of asserting that there exists a selection of equilibria across different problems that is well defined, and satisfies both replication consistency and implementation consistency. For that matter, a bargaining problem is a *replicated problem* if it is a $k$-replication (for some $k \in \mathbb{N}$) of another bargaining problem. And it is a *replaced problem* if there are $n-1$ identical, linear preferences out of its $n$ involved preferences, where the vector of coefficients characterizing these linear preference is an equilibrium price vector in the exchange economy corresponding to this problem. Note that if agents in a problem are replaced by a linear preference characterized by an equilibrium price vector for that problem, then the same equilibrium price vector prevails in the resulting replaced problem.

For any bargaining problem which is not replicated nor replaced, the selection of an equilibrium price vector is unconstrained. Any equilibrium price vector may be selected, and all of its corresponding equilibrium allocations are assigned as a resolution. Given that selection, Replication Invariance (**A4**) dictates the selection in replicated problems. Impartiality (**A3**), through the requirement that the equilibrium price vector as a linear agent can implement the solution, dictates the selection in replaced problems.

Finally, the above choices are well-defined as any single replicated or replaced problem cannot be obtained from two different problems with different equilibria selections. Replicated problems can always be traced to the maximal number of replications, and by Replication Invariance (**A4**), the same goes for the choice in any other replicated problem along the way. With regard to replaced problems, whenever the number of agents is at least three a replaced problem can only originate from one root problem. A problem may arise only in problems involving two agents with differing linear preferences, in case both characterizing coefficients of these preferences constitute equilibrium price vectors for the implied exchange economy. However it is well known that with linear preferences (under our structural assumption A0) equilibrium is unique. Therefore, such a situation cannot obtain and the selection described above is well defined. Moreover, it satisfies both replication consistency and implementation consistency.

### 4.1.2 Proving that (ii) implies (i)

Suppose that (ii) of the theorem holds. Assumption **A1** readily obtains since all equilibrium allocations are Pareto optimal.

In order to prove **A2**, observe first that all the allocations corresponding to the same vector of equilibrium prices are indifferent for all agents. For the other part of **A2**, suppose that $a = (a(i))_{i=1}^{n}$ and $a' = (a'(i))_{i=1}^{n}$ are two allocations between which all agents are indifferent: $a(i) \sim^i a'(i)$ for every $i$. Assume that $a$ is an equilibrium allocation corresponding to an exchange economy with equal initial endowments for all. Consequently, according to the proof of Claim 1, $a'$ is an equilibrium allocation for the same prices, and thus either both are assigned by a solution or none is.

Impartiality (**A3**) is implied since for any equilibrium allocation the corresponding vector of equilibrium prices can be extended to a linear preference relation, which is unprejudiced as a simple implication of its definition, and assigns the same value to all the bundles in each equilibrium allocation. The same equilibrium prices prevail in the original exchange economy and in the exchange economy constructed by any one of the original agents and $n-1$ copies of the linear prices-agent, when all are equally endowed. By implementation consistency the selected equilibria both in the original problem and in any replaced problem are the same. Thus, the same allocations are assigned by a solution.

Part (a) of Replication Invariance (**A4**), which states that replications of allocations in a solution are included in the solution to the replicated problem, follows from the assumption that a solution is replication-consistent. This restricts a solution to choose the same market equilibrium for both the replicated and the original problems.

To prove part (b) of Replication Invariance let $(x, (\succsim^i)_{i=1}^n)$ be a bargaining problem and $a$ an allocation in its solution. Suppose that there exists an unprejudiced relation $\succsim^*$, that conforms to A0(2), perceives $a$ as fair and can implement it in the designated problem, through $\varphi$. We show that in consequence $\succsim^*$ must be a linear preference characterized by the vector of equilibrium prices that corresponds to $a$.

The fact that an unprejudiced relation is a linear relation results from Lemma 1. Let $m = (m_1, \ldots, m_L)$ be a vector of coefficients characterizing $\succsim^*$, normalized so as to satisfy $m \cdot x = n$. Since $a$ is perceived as fair by $\succsim^*$ all bundles $a(i)$ are $\succsim^*$-indifferent to $x/n$, hence $m \cdot a(i) = 1$ for every $i$.

Let $p = (p_1, \ldots, p_L)$ denote the vector of equilibrium prices corresponding to the allocation under consideration, and again normalize it by $p \cdot x = n$. Since a solution assigns equilibrium allocations that correspond to an equilibrium under equal endowments for all, $p \cdot a(i) = 1$ for every agent $i$.

Given a good $\ell$, let $i$ be an agent such that $a_\ell(i) > 0$ and examine the replaced problem in which $\succsim^*$ replaces all agents but $i$. Denote the equilibrium prices in this replaced problem by $p^i$, where $p^i$ is again normalized so that $p^i \cdot x = n$, hence $p^i \cdot a(j) = 1$ for every $j$. Since it was assumed that $a$ can be implemented by $\succsim^*$ in the problem $(x, (\succsim^i)_{i=1}^n)$, through $\varphi$, $a$ is included in the solution to this replaced bargaining problem. According to assumption **A0** (and specifically its part (e)), since agent $i$ consumes the same bundle $a(i)$ in this replaced problem, and since this agent is still subject to the same budget constraint, the fact that $a_\ell(i) > 0$ implies $p^i_\ell = p_\ell$. Furthermore, each copy of $\succsim^*$ consumes a bundle $a(j)$, for $j \neq i$. Linearity of $\succsim^*$ implies that if $p^i_r = 0$ then necessarily also $m_r = 0$. Otherwise, for any good $r$ with a positive price, for which $a_r(j) > 0$ for some $j \neq i$, it holds that $m_r/p^i_r = t$ for the same number $t$. According to our normalization, however, for every such $j$, $1 = m \cdot a(j) = tp^i \cdot a(j) = t$ (all the goods not consumed by $j$ under $a$, as well as any goods with zero price, contribute zero to this sum), therefore $m_r = p^i_r$ for any good $r$ that is consumed by any agent other than $i$ under $a$. On the other hand, in order for $\succsim^*$ *not* to consume goods that are not consumed by $j \neq i$ under $a$ it should hold that for any such good $r$, $m_r \leq p^i_r$. In any case, for good $\ell$ it holds that $m_\ell \leq p^i_\ell = p_\ell$.

Now the equality $m \cdot x = p \cdot x$ is employed, which together with the assumption $x >> 0$ yields the desired result: $m_\ell = p_\ell$ for any good $\ell$. In summary, whenever an equilibrium allocation in the exchange economy corresponding to a bargaining problem is perceived as fair and can be implemented by an unprejudiced relation $\succsim^*$, conforming to A0(2), then $\succsim^*$ is a linear relation characterized by the vector of coefficients $p$, $p$ being the vector of equilibrium prices corresponding to that allocation.

Replication consistency and implementation consistency guarantee that the same equilibrium price vector is chosen for $(x, (\succsim^i)_{i=1}^n)$ and for any replaced problem created from its $k$-replication by replacing one agent with $\succsim^*$. Therefore, under the chosen equilibrium prices for these replaced problems, $\succsim^*$ is indifferent between all bundles that exhaust the budget constraint. Since, as already established, any allocation contained in the solution to the original problem is also contained in the solution to its $k$-replication, and as any such allocation exhausts the budget constraint, any allocation in the solution to the original problem is contained in the solution to any replaced problem created from the replicated problem. Namely, $\succsim^*$ can implement $a^k$ in the $k$-replicated problem, through $\varphi$.

### 4.2 Proof of Theorem 2

Let $\varphi$ be a bargaining solution to (possibly asymmetric) problems that conform to the structural assumption **A0d**, and suppose that $\varphi$ satisfies assumptions **A1,A2,A3d** and **A4**.

The proof for bargaining problems with a general disagreement point $d$ is very similar to the proof of Theorem 1, which addresses the case of an all-zeros disagreement point. Denote a disagreement point by $d = (d(1), \dots, d(n))$ with $d(i) = (d_1(i), \dots, d_L(i))$ being the bundle allocated to agent $i$ in case of disagreement. Consider a bargaining problem $(x, (\succsim^i)_{i=1}^n, d)$, satisfying **A0d**, and let $a$ denote an allocation in its solution. By Impartiality (**A3d**) there exists an unprejudiced relation $\succsim^*$, that conforms to A0(2), which perceives $a$ as fair given $d$, and can implement it in the problem $(x, (\succsim^i)_{i=1}^n, d)$, through $\varphi$. According to Lemma 1, $\succsim^*$ is a linear preference. Namely, there is a vector of coefficients $m = (m_1, \dots, m_L)$ such that for any two bundles $y$ and $z$, $y \succsim^* z$ if and only if $m \cdot y \geq m \cdot z$. As $\succsim^*$ is monotone, $m \geq 0$.

The linear preference $\succsim^*$ perceives $a$ as fair given $d$. Namely, there exists a bundle $t \in \mathbb{R}^L_+$ such that $a(i) \sim^* d(i) + t$ for all $i$. Simple arithmetics yields that $m \cdot t = m \cdot \dfrac{(x - \sum_{j=1}^n d(j))}{n}$. Therefore, $m \cdot a(i) = m \cdot d(i) + m \cdot \dfrac{(x - \sum_{j=1}^n d(j))}{n}$ for every $i$. Denote $w = m \cdot \dfrac{(x - \sum_{j=1}^n d(j))}{n}$.

Similarly to the proof of the symmetric case, consider the bargaining problem $(x, (\succsim^i, \underbrace{\succsim^*, \dots, \succsim^*}_{n-1 \text{ times}}), d)$, created when all agents $j \neq i$ are replaced by $\succsim^*$, where this time a copy of $\succsim^*$ that replaces agent $j$ also inherits $j$'s disagreement bundle $d(j)$. According to Impartiality (**A3d**), the allocation $a$ is contained in the solution to this replaced problem. Therefore, a copy of $\succsim^*$ that replaces an agent $j \neq i$ gains a utility value of $m \cdot d(j) + w$ in the solution to this replaced problem. Let $y \in \mathbb{R}^L_+$ be a bundle that satisfies $x \geq y$ and $m \cdot y - m \cdot d(i) =$

$m \cdot \frac{(x - \sum_{j=1}^{n} d(j))}{n} = w$. Note that in this case it holds that $m \cdot (x - y) = (n-1)w + \sum_{t \neq i} m \cdot d(t)$. Define:

$$\lambda = \frac{(n-1)w}{(n-1)w + \sum_{t \neq i} m \cdot d(t)}$$

$$\theta_j = \frac{m \cdot d(j)}{\sum_{t \neq i} m \cdot d(t)} \ , \quad j \neq i$$

Consider the distribution of $x$ that assigns $y$ to agent $i$ and assigns $\dfrac{\lambda(x-y)}{n-1} + \theta_j(1-\lambda)(x-y)$ to the copy of $\succsim^*$ that replaces agent $j \neq i$. Multiplying each of these bundles, for $j \neq i$, by $m$ delivers $w + m \cdot d(j)$. A copy of the relation $\succsim^*$ that replaces agent $j \neq i$ is therefore indifferent between the bundle under this allocation and the bundle $a(j)$. The Pareto assumption implies that $a(i) \succsim^i y$. It follows that $a(i) \succsim^i y$ for any bundle $y$ that satisfies $x \geq y$ and $y \cdot m = d(i) \cdot m + w$.

In the same manner as in the proof of Theorem 1, the double problem is next invoked and the proof continues as above, yielding eventually that the solution to a problem $(x, (\succsim^i)_{i=1}^n, d)$ assigns allocations $a$ that satisfy the following: there is a vector $m = (m_1, \dots, m_L)$ and a bundle $a(i)$ for each agent $i$ such that $m \cdot a(i) = m \cdot d(i) + w$, and $a(i)$ is weakly preferred by agent $i$ to any other bundle satisfying the same constraint. This is exactly the formulation of the constrained optimization problem of a pure exchange economy where each agent $i$ is initially endowed with $d(i) + \dfrac{1}{n}\left(x - \displaystyle\sum_{i=1}^{n} d(i)\right)$. Again, the assumptions on the preferences and on the disagreement point indicate the existence of an equilibrium. The same selection as above can be applied and the result follows: a solution assigns to each bargaining problem $(x, (\succsim^i)_{i=1}^n, d)$ all the equilibrium allocations that correspond to one vector of equilibrium prices, while maintaining replication consistency (the same equilibrium is chosen for a problem and its replications) and implementation consistency (the same equilibrium is chosen for a problem and its replacements by the linear preference characterized by the equilibrium price vector).

In order to prove that (ii) of the theorem implies (i), the only difference compared to the proof of Theorem 1 lies in proving part (b) of Replication Invariance. For that matter, suppose a bargaining problem $(x, (\succsim^i)_{i=1}^n, d)$ and an equilibrium allocation $a$ in its solution. Denote by $p = (p_1, \dots, p_L)$ the vector of equilibrium prices corresponding to $a$, and normalize it by $p \cdot x = n$. Recall that a solution to a dispute contains equilibrium allocations under endowments $d(i) + \dfrac{(x - \sum_k d(k))}{n}$ per agent $i$. Thus, for every agent $i$, $p \cdot a(i) = p \cdot d(i) + p \cdot \dfrac{x - \sum_k d(k)}{n}$.

Let $\succsim^*$ be an unprejudiced binary relation which perceives $a$ as fair given $d$ and can implement it in the problem under consideration, through $\varphi$. It should be proved that for any $k \in \mathbb{N}$, $\succsim^*$ can implement the allocation that is a $k$-replication of $a$ in the bargaining problem that is a $k$-replication of the problem under consideration. Once again, Lemma 1 implies that the relation $\succsim^*$ under consideration is linear. Denote its coefficients by $m = (m_1, \ldots, m_L)$, normalized so that $m \cdot x = n$. Fairness of $a$ given $d$ according to $\succsim^*$ means that $(m \cdot a(i) - m \cdot d(i))$ is constant across $i$. Therefore, by summing over $i$, $m \cdot a(i) = m \cdot d(i) + m \cdot \dfrac{x - \sum_k d(k)}{n}$.

Given a good $\ell$, let $i$ be an agent such that $a_\ell(i) > 0$. Now examine the replaced problem in which $\succsim^*$ replaces all agents but $i$ (along with their disagreement bundles). Under (b) of Replication Invariance it is assumed that $a$ can be implemented by $\succsim^*$ in the problem $(x, (\succsim^i)_{i=1}^n, d)$. Thus, $a$ is included in the solution to this replaced problem. According to assumption **A0** (and specifically in its part (e)), since agent $i$ consumes the same bundle $a(i)$ in this replaced problem, and as this agent is still subject to the same budget constraint, the fact that $a_\ell(i) > 0$ implies that $p^i_\ell = p_\ell$. Denote the vector of equilibrium prices in this replaced problem by $p^i$, where $p^i$ is again normalized so that $p^i \cdot x = n$. A solution assigns equilibrium allocations that correspond to endowments of $d(j) + \dfrac{(x - \sum_{k=1}^n d(k))}{n}$ per agent $j$. Therefore, here as well it holds that $p^i \cdot a(j) = p^i \cdot d(j) + p^i \cdot \dfrac{x - \sum_k d(k)}{n}$ for every $j$.

For $j \neq i$, linearity of $\succsim^*$ that replaces $j$ implies that whenever $p^i_r$ is zero then so is $m_r$. Otherwise, there exists a number $t$ such that for any good $r$ with a non-zero price, for which $a_r(j) > 0$, it holds that $m_r/p^i_r = t$, and for any good $r$ for which $a_r(j) = 0$, $m_r/p^i_r \leq t$. Consequently, $m \cdot a(j) = tp^i \cdot a(j)$ for $j \neq i$. Combining this with the above equalities that pertain to $m$ and to the equilibrium price vectors, it follows that for $j \neq i$, $m\left(d(j) + \dfrac{(x - \sum_{k=1}^n d(k))}{n}\right) = m \cdot a(j) = tp^i \cdot a(j) = tp^i\left(d(j) + \dfrac{(x - \sum_{k=1}^n d(k))}{n}\right)$. Namely, $(m - tp^i)\left(d(j) + \dfrac{(x - \sum_{k=1}^n d(k))}{n}\right) = 0$. As $m - tp^i \leq 0$ and $d(j) + \frac{(x - \sum_{k=1}^n d(k))}{n} >> 0$ it must be that $m_r = tp^i_r$ for every good $r$. The identical normalization of $m$ and $p^i$ delivers $t = 1$. That is, $m_r = p^i_r$ for every $r$, and specifically, $m_\ell = p^i_\ell = p_\ell$. Repeating the same arguments for every $\ell$ yields that $m_\ell = p_\ell$ for every good $\ell$, and the proof continues as in Theorem 1.

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