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The Benefit of Collective Reputation*

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Abstract

We study a model of reputation with two long-lived firms who operate under a collective brand or as two individual brands. Firms' investments in quality are unobserved and can only be sustained through reputational concerns. In a collective brand, consumers cannot distinguish between the two firms. In the long run, this generates incentives to free-ride on the other firm's investment, but in the short-run, it mitigates the temptation to milk a good reputation. The signal structure and consumers' prior beliefs determine which effect dominates. We interpret our findings in light of the type of industry in which the firms operate.

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1 Introduction

Firms make substantial investments to build strong brands. The American Marketing Association defines a brand as "a name [...] that identifies one seller's good [...] as *distinct* from those of other sellers."[1] Sometimes, firms sell their products under a shared name or a collective brand that carries a collective reputation that is shaped by the firms who bear the name. For example, a bottle of wine carries an appellation, such as Bordeaux or Champagne, which applies to many producers in the same region. Many lay consumers cannot distinguish among the names of individual producers and rely on appellations to make their purchase decisions. Country of origin labelling serves a similar function. For example, Volkswagen advertises "the power of German engineering" and Swiss watchmakers, even the ones with strong individual brands, emphasize that their watches are "Swiss made."

Collective reputation building is also relevant in other domains. Any physical good that is purchased online is essentially an "experience good" whose quality cannot be ascertained by consumers at the time of purchase (Nelson, 1970).[2] Nosko and Tadelis (2015) show that a consumer who has a bad experience with one seller in an online platform, such as eBay or Amazon, is less likely to buy through that platform again, which is evidence of a "reputational externality" that sellers in the platform exert over one another. Such a reputational externality is characteristic of a collective brand.[3]

Both individual and collective brands are means to build a good reputation. When building a reputation, a firm faces a moral hazard problem; its investment in quality is unobservable to current consumers, and the reputational return on its investment can only be collected in the future. In this article, we study how sustaining reputation in a collective brand is different from that of an individual brand.[4]

[1] https://www.ama.org/resources/pages/dictionary.aspx

[2] Experience goods include nondurables such as wine, durables such as appliances and cars, and many service providers such as lawyers, doctors, and mechanics.

[3] Another example for the reputational externality is provided by organizations that require their members to wear uniforms, such as the police, military forces, and Girl and Boy Scouts. Uniforms foster the creation of a reputational externality among their wearers because they blur individual identities.

[4] In practice, firms are often endowed with features of both individual and collective brands. For example,

At first glance, collective brands may seem like a bad idea. If several firms operate under one brand name, each firm has an incentive to free-ride on other firms' investments. Moreover, a firm's investment in its quality has a weaker effect on the brand value of a collective brand because consumers are uncertain about the relationship between the collective brand's reputation and the specific firm with which they interact. In other words, the "precision" of the signal that is generated by a firm's investment in quality is lower in a collective brand, which weakens the incentive to invest in quality.

Nevertheless, under some circumstances, a collective reputation can serve as a commitment device for investment in high quality. If a brand is already very successful, then a firm might be discouraged from additional investment because the *short run* return from it is small. Such a firm might become complacent or rest on its laurels. Analogously, if a brand develops a bad reputation, then additional investment may be insufficient to improve its reputation in the short run. Collective brands can mitigate these short run "discouragement effects" faced by individual firms after very good or very bad histories by making extreme beliefs about the value of the brand less likely.

However, a member of a collective brand may also be tempted to free-ride on future efforts by other members of the brand, which is irrelevant in the case of an individual brand. So, in the long-run, a collective brand provides weaker investment incentives than an individual brand. It follows that when short-run incentives are more important, then a collective brand can provide stronger incentives to invest than an individual brand, and when long-run incentives are more important then the opposite result holds.[5] Below, we describe specific conditions where the short run benefit of a collective brand outweighs the long-run benefits of an individual brand.

To compare the two branding regimes, we consider a model in which two or more long-

Volkswagen belongs to the group of German automakers, and also possesses a strong individual brand. We abstract from such hybrid situations to present the difference between individual and collective brands in the starkest terms.

[5]As shown in Section 4 below, in some circumstances, collective brands dominate individual brands even if firms are infinitely patient.

lived firms sell their products over time under an individual or a collective brand. There are two types of firms, competent and incompetent. In every period, competent firms can make a costly investment to increase the probability of producing good quality in that period. Incompetent firms lack this ability. Firms' investments are unobservable to consumers. Consumers observe past quality realizations, which are noisy signals of firms' investments and competence.

The critical distinction between an individual and a collective brand lies in consumers' observation of past quality realizations. Consumers observe a firm-specific record under an individual brand and a group-specific record under a collective brand. This has two implications. First, each signal produced by a collective brand is noisier because, unlike in the case of an individual brand, consumers facing a collective brand cannot trace the signal back to the firm that produced it. Second, a collective brand generates more signals than an individual brand because each one of its members can produce a signal.

In both cases, we focus on the most efficient equilibrium in which a competent type always invests. We call this equilibrium the "reputational equilibrium." We examine whether it is "easier" for an individual or a collective brand to sustain this reputational equilibrium in the sense that the equilibrium can be sustained for a broader set of parameter values.

We focus the analysis on two extreme cases: the case where the observation of good quality indicates competence, which we call "Exclusive knowledge;" and the case where the observation of bad quality indicates incompetence, which we call "Quality control." In the former case, specialized exclusive knowledge is required in order to produce high quality (as in the case of expensive watches and cars); in the latter case, investment is needed to perform good quality control, so the production of low quality reveals incompetence (as in the case of mass products and generics). This emphasis has two reasons. It simplifies the analysis, and it allows us to focus on clearly identified circumstances where collective reputation is superior to individual reputation.

We show that a collective brand can support a reputational equilibrium for higher investment costs than individual brands in "exclusive knowledge" markets when the ex-ante probability of competence is high. Individual brands perform poorly in such markets because it is relatively easier for them to establish an excellent reputation quickly (as the production of high quality reveals competence), which they can then milk. The opposite result (namely, a collective brand supports a reputational equilibrium for higher investment costs than individual brands when the ex-ante probability of competence is low) holds in "quality control" markets where production of low quality reveals incompetence.

The benefits from the additional commitment power that is provided by a collective brand can be significant enough to induce a competent firm to form a collective brand with an incompetent firm voluntarily. In such a case, the socially optimal branding regime coincides with firms' optimal choice, so no regulation is required. However, it is also possible that a competent firm would prefer an individual to a collective brand, even though the latter induces incentives to invest and the former does not. In such cases, a regulation that promotes collective brands would improve social welfare.

The article is structured as follows. The next section discusses the related literature. In Section 3 we present the model, define the equilibrium concept, and introduces the critical distinction between an individual and a collective brand in terms of consumers' beliefs. In Section 4 we describe circumstances under which an individual or a collective brand provides stronger incentives for investment. In Section 5, we examine a competent type's brand formation decision and consider whether it would want to form a collective brand with an incompetent firm. In Section 6 we present extensions of the basic model that allow for a longer memory and more than two firms, respectively. Section 7 concludes. All proofs are relegated to Appendix A and Appendix B.

2 Related Literature

Our work is related to the theoretical economics literature on reputation in markets for experience goods, as well as to the literature on umbrella branding, country-of-origin and career concerns.

The idea that reputational concerns may induce a firm to produce high quality even though consumers are unable to verify the quality of an experience good goes back to Klein and Leffler (1981). The subsequent literature has explored the implications of this argument and has argued that it must contend with two major difficulties: first, for reputation to be sustainable, it must generate profits, and it is not clear how this is possible in a competitive environment where profits are driven down to zero (See Kranton, 2003).[6] The second difficulty, which has been famously noted by Holmström (1999), is that in the long run, the firm would develop an excellent reputation for quality. Any observation of low quality would thus be attributed to bad luck and would therefore not affect prices, with the consequence that the firm's incentive to continue to exert the costly effort necessary to produce high quality would be destroyed.

Several elegant solutions to these difficulties were offered. Hörner (2002) noted that if consumers can observe the consumer bases of firms in the market, then a firm may be discouraged from producing low quality for fear of losing its consumer base (see also Fedele and Tedeschi, 2014). Another solution was suggested by Mailath and Samuelson (2001), who formulated the insight that individual reputation can be sustained if consumers' beliefs about the type of the firm are bounded away from one, as would be the case if the firm's unobserved type is drawn afresh in every period. In this article, we assume instead that consumers have finite memories as in Moav and Neeman (2010) and Liu and Skrzypacz (2014). The benefit of this assumption is that it allows us to explicitly solve for the threshold cost below which firms invest in quality, which is an intractable problem in Mailath and Samuelson (2001)'s

[6]In our model we abstract away from this difficulty by assuming that firms make take-it-or-leave-it price offers to consumers who are each matched with only one firm, but our results would continue to hold as long as firms capture at least some of the surplus that is generated by their transactions with consumers.

model.

Using this framework we can show that acting as a collective can help sustain a high reputation. The mechanism is related to the one studied by Bar-Isaac (2007) who considers an overlapping generations model in the moral hazard-in-teams (career concerns) framework developed by Holmström (1999). He shows that senior entrepreneurs who sell the firm in the next period have an incentive to exert effort and work with young juniors who themselves also need to build a good reputation.

Research that identifies the benefits of collective reputation is scarce. Tirole (1996) is probably the first who formalized an analytical model of collective reputation in the context of a large organization. In Tirole's model, a group's reputation is an aggregate of the reputations of the individual members of the group. As is the case in models with statistical discrimination, there can be different steady states equilibria, and in particular one with "low corruption" and another with "high corruption."[7] Unlike Tirole (1996), our focus is on the moral hazard problem and "brand management" rather than on statistical steady-state inferences.

More recently, Fishman, Finkelstein, Simhon, and Yacouel (2018) consider a two-period model in which an individual firm can only generate one signal per period. A collective brand that includes many firms can send many signals and so provide better information to consumers. This informational benefit outweighs firms' incentive to free-ride on other firms' investment efforts provided the number of brand members is not too large.[8] Their model abstracts away from the issue of commitment and the dynamic trade-offs that are the focus of our analysis. Fleckinger (2016) considers collective reputation under Cournot competition where consumers only learn the average quality in the market. He studies the effect of the number of firms on welfare and shows that quality is decreasing in the number of firms whereas quantity increases. Fleckinger, Mimra, and Zago (2017) consider a static

[7]Levin (2009) extends Tirole (1996) by considering the case where the cost of high effort evolves stochastically over time.

[8]Indeed, Winfree and McCluskey (2005) claim that a large number of apple growers in the state of Washington contributed to the decline in the quality of Washington apples during the 1990s.

model of collective reputation where all those producers who failed to pass an inspection have the same collective reputation. In this model, collective reputation is shown to possibly yield higher quality and welfare than individual reputation, as inter-producer free-riding under full collective reputation might be less severe than intra-producer free-riding under individual reputations. Moreover, unlike Bar-Isaac (2007), Tirole (1996), Fleckinger (2016) and Fishman, Finkelstein, Simhon, and Yacouel (2018), our focus is on the comparison between individual and collective reputation.

Finally, collective reputation has also been studied in the context of umbrella branding, where an existing brand name is extended to a new product line, and so the reputation of the brand depends on the performance of the various products included under it. Wernerfelt (1988), Choi (1998), Cabral (2000), Miklós-Thal (2012), and Moorthy (2012) have examined the incentives that a monopolist has to signal quality by pooling reputation for different products. In a model with moral hazard, where consumers' can perfectly monitor product quality, Andersson (2002) and Hakenes and Peitz (2008) show that umbrella branding always provides stronger incentives to invest because any deviation jeopardizes the firm's reputation in many markets. Yu (2018) examines the extent of risk sharing across product markets as a function of the relatedness between markets and shows that independent branding can be a disciplinary device if relatedness is high. Others have considered settings where free-riding incentives are more pronounced. Zhang (2015) examines country-of-origin labelling. He shows that the ability to free-ride on other firms' quality investments implies that high-quality firms have an incentive to dissociate themselves from the country-of-origin label, which in turn mitigates free-riding and can improve the reputation for the group.

3 Model and Definitions

Model

We consider a market with two firms that produce a vertically differentiated experience good, that can be of either good (G) or bad (B) quality. In every period, $t \in \{\dots, -1, 0, 1, \dots\}$, one short-lived consumer with unit demand arrives and is randomly matched with one of the two firms.

Firms. Each firm is competent (C) with probability $\mu \in (0,1)$, or incompetent (I) with probability $1-\mu$. The two firms' types $\theta \in \{C, I\}$ are stochastically independent. The firms' types are commonly known between the firms but are unobservable by the consumers. Firms' types remain fixed throughout the duration of the game. A competent firm that is matched with a consumer can make an investment at cost $c > 0$ to improve the quality of the good it produces in that period: investment yields a good quality (G) with probability $\pi_H \in (0,1]$ whereas a lack of investment yields good quality with probability $0 \leq \pi_L < \pi_H$. An incompetent firm cannot invest and produces good quality with probability π_L. We assume that $\Delta\pi \equiv \pi_H - \pi_L \geq c$ so that investment is always socially optimal.

Consumers. Consumers do not observe the firms' investment decisions, but they do observe the quality of goods produced in the last two periods.[9] Consumers update their beliefs about the type of the firm they are matched with based on past observations of quality.

Timing. After its investment decision, the firm makes a take-it-or-leave-it offer to the consumer with whom it is matched. We focus on the case in which this offer is equal to the consumers' posterior willingness to pay for the good conditional on her beliefs, and is independent of the realized quality of the produced good. The consumer either accepts or rejects the firm's offer and then leaves the market.

[9] In Section 6, we allow consumers to observe quality realizations of the past $2 \leq T < \infty$ periods and show that the main results are robust. For $T = 1$ an individual brand always provides stronger incentives for effort than a collective brand.

Payoffs. We normalize the payoff of a consumer who does not buy the good to 0. A consumer who buys the good at a price p receives a payoff of $1 - p$ or $-p$ if the good is of good or bad quality, respectively. A firm that sells in period t at price p_t obtains a payoff of $p_t - c$ at t if it invests at t, and p_t if it does not. A firm that does not sell in any given period obtains a payoff of 0 in that period. Firms discount their future payoffs by $\delta \in [0, 1)$.

Branding Regimes. In a collective brand, consumers cannot distinguish between the identities of the two firms. This means that consumers obtain a signal about the collective brand in every period, regardless of which firm they are matched with. Thus, the set of relevant histories for a collective brand is $\{G, B\} \times \{G, B\}$. In contrast, if firms form individual brands, then consumers can distinguish between them. Consequently, consumers observe the quality produced in the last two periods by the firm with which they have been matched. Thus, the set of relevant histories for an individual brand is $\{G, B, \varnothing\} \times \{G, B, \varnothing\}$ where $\varnothing$ represents a failure of the firm to match.

We denote a history at time t by $\mathbf{h}_t \equiv h_{t-2}h_{t-1}$ where h_{t-n} denotes the quality of the good produced in period $t - n$. Notice that the matching process ensures that the two firms sell the same expected quantity under the two regimes, but at possibly different prices.

Equilibrium. We focus on *Markov equilibria* in which strategies depend only on the relevant histories specified above. A stationary equilibrium is defined by an investment and pricing strategy of firms, a purchasing strategy of consumers, and consumers' beliefs over the type of the firm they are matched with. Firms' and consumers' strategies are best responses to each other and to each future self that is playing the proposed equilibrium strategy, given consumers' beliefs, and consumers update their beliefs based on past observations using Bayesian updating whenever possible.

Observe that in equilibrium consumers must purchase the good when indifferent because if they don't then firms' best responses are not well defined.

Beliefs and Signal Structure

Posterior beliefs. In the case of an individual brand, posterior beliefs given a history $\mathbf{h}_t$ are given by the probability $\Pr^{\text{ind}}(\mathbf{h}_t)$ that the firm the consumer is matched with is competent.

In the case of a collective brand, this posterior is formed based on beliefs about the distribution of pairs of types of the two firms in the collective. We denote the posterior belief that the two firms' types are $s \in \{C, I\}^2$ given history $\mathbf{h}_t$ by $\eta_s(\mathbf{h}_t)$. The posterior belief that the matched firm is competent given a history $\mathbf{h}_t$ is

$$\Pr^{\text{col}}(\mathbf{h}_t) = \eta_{CC}(\mathbf{h}_t) + \frac{1}{2}(\eta_{CI}(\mathbf{h}_t) + \eta_{IC}(\mathbf{h}_t)). \tag{1}$$

The *reputation* of a brand – both individual and collective – corresponds to the two posterior beliefs $\Pr^{\text{ind}}(\mathbf{h}_t)$ and $\Pr^{\text{col}}(\mathbf{h}_t)$, respectively.

For most of the article we focus our attention on the stationary equilibrium in which competent firms invest in quality whenever they are matched with a consumer, after each history, and independently of the other firm's type. We call this the **reputational equilibrium**. In such an equilibrium, upon observing a history $\mathbf{h}_t$, a consumer is willing to pay a price

$$p^b(\mathbf{h}_t) = \Pr^b(\mathbf{h}_t) \cdot \pi_H + (1 - \Pr^b(\mathbf{h}_t)) \cdot \pi_L, \tag{2}$$

where $b \in \{\text{ind}, \text{col}\}$. Thus, this is also the reputational equilibrium price. Our assumption that $\Delta\pi \geq c$ implies that the reputational equilibrium is socially optimal.[10]

The game also has other stationary equilibria. For example, a "no investment" equilibrium, in which a competent firm never invests in quality, always exists. We discuss different stationary equilibria in Section 5 where we address endogenous brand formation.

As mentioned in the introduction, when we compare the incentives induced by collective

[10]We assume that if a firm charges a higher price, then consumers update beliefs about the brand's type as if it has just produced a bad outcome.

versus individual brands, we focus on two types of signal structures that are easy to interpret and that highlight the benefit of collective reputation:

1. *"Exclusive knowledge" ($\pi_L = 0$, $\pi_H \in (0,1)$):* In this case, a firm cannot produce a good outcome without making an investment. Consequently, the observation of good quality reveals competence. Such a signal structure fits industries in which some special technology or expertise is required to produce high-quality products, such as in the case of watches, automobiles and electronics.

2. *"Quality control" ($\pi_H = 1$, $\pi_L \in (0,1)$):* In this case, a competent firm that invests is guaranteed a good outcome. Consequently, observation of bad quality reveals incompetence (in the reputational equilibrium). Such a signal structure fits industries in which perfect quality control is necessary to produce high-quality products consistently, such as in manufacturing or service industries.

We analyze the game for general signal structures. However, to derive economically meaningful results, we focus on the case of exclusive knowledge when comparing individual and collective brands. We discuss the analogous results for the case of quality control in the text and provide the related formal findings in Appendix B. Note that by continuity, all of our results for the case of exclusive knowledge also hold in the case where π_L is sufficiently close to zero and the value of π_H is held fixed, and all of our results for the case of quality control also hold in the case where π_H is sufficiently close to one and π_L is held fixed.

4 Reputational Equilibrium

In this section, we derive necessary and sufficient conditions for the existence of the reputational equilibrium under the two branding regimes. We show that a reputational equilibrium exists if and only if the investment cost c is smaller than or equal to a threshold cost $\bar{c}^b$, $b \in \{\text{ind}, \text{col}\}$, that depends on the branding regime. Then, we identify which branding regime sustains the reputational equilibrium for a larger set of costs c by comparing these two threshold costs. The branding regime with a higher threshold cost $\bar{c}^b$ is said to induce stronger incentives to invest.

Individual Brand

In a reputational equilibrium, a competent firm invests in quality after a history $\mathbf{h}_t$ if its expected return from an investment, taking into account the effect of this investment on the consumers' future willingness to pay, is higher than its cost. The first proposition characterizes the threshold cost of investment above which an investment is not worthwhile.

Proposition 1. *The reputational equilibrium exists for an individual brand if and only if the cost of investment c is such that*

$$c \leq \bar{c}^{ind} \quad \equiv \quad \min_{h_{t-1} \in \{G,B,\varnothing\}} \bar{c}^{ind}(h_{t-1})$$

where $\bar{c}^{ind}(h_{t-1})$ denotes the expected benefit from investment after history $\mathbf{h}_t = h_{t-2}h_{t-1}$. The function $\bar{c}^{ind}(h_{t-1})$ is given by

$$\begin{aligned} \bar{c}^{ind}(h_{t-1}) \equiv \quad & \tfrac{\Delta\pi}{2} \cdot \delta \cdot \big[\big(p^{ind}(h_{t-1}G) - p^{ind}(h_{t-1}B) \big) + \\ & \delta \cdot \textstyle\sum_{h_{t+1} \in \{G,B,\varnothing\}} \pi(h_{t+1}) \cdot (p^{ind}(Gh_{t+1}) - p^{ind}(Bh_{t+1})) \big] \end{aligned} \tag{3}$$

where $\pi(h_{t+1})$ denotes the probability distribution of the outcome realized in period $t+1$ ($\pi(G) = \frac{\pi_H}{2}, \pi(B) = \frac{1-\pi_H}{2}$, and $\pi(\varnothing) = \frac{1}{2}$).

The threshold cost $\bar{c}^{\text{ind}}$ is the sum of expected short-run and long-run price premiums that arise from an investment, as explained below. All these price premiums can be expressed explicitly as a function of the parameters of the model, μ, π_H and π_L.[11] We relegate the explicit expression to Appendix A because it is lengthy and not insightful in itself.

[11] This is because, in the reputational equilibrium, probabilities and prices can be explicitly calculated for every history. For example, $\pi^{\text{ind}}(GG) = \frac{\mu\pi_H^2}{\mu\pi_H^2 + (1-\mu)\pi_L^2}$ and then $p^{\text{ind}}(GG)$ can be calculated using (2). The other probabilities and prices can be calculated in a similar way.

The firm's investment in period t increases the probability of producing a good outcome at t, and this will be observed by the consumer matched with the firm in the next two periods $t+1$ and $t+2$. Upon observing $h_t = G$, such a consumer would be willing to pay more than if she observed $h_t = B$. So, the threshold cost is the sum of expected price premiums in the following two periods. The differences in expected price premiums in periods $t+1$ and $t+2$ induce *short-run* and *long-run* incentives to invest, respectively.

In the short-run, a consumer that is matched with the firm in period $t+1$ observes the history $\mathbf{h}_{t+1} = h_{t-1}h_t$. So, by investing in period t, the firm enjoys a price premium $p^{\text{ind}}(h_{t-1}G) - p^{\text{ind}}(h_{t-1}B)$. This premium is small if the firm has a very good or very bad reputation following the history h_{t-1}. For example, in the exclusive knowledge environment, following the history $h_{t-1} = G$, a the consumer's posterior belief is updated to $\Pr^{\text{ind}}(h_{t-1}h_t) = 1$. Thus, for $h_{t-1} = G$, the short-run price premium vanishes, or $p^{\text{ind}}(h_{t-1}G) - p^{\text{ind}}(h_{t-1}B) = 0$. This illustrates the difficulty of inducing a commitment to invest through short-run incentives for an individual brand. An individual brand can develop a very good reputation through investment, but it is then tempted to exploit its reputation.

In the long-run, the consumer that is matched with the firm in period $t+2$ no longer observes the original history ($\mathbf{h}_t$). Instead, she observes $\mathbf{h}_{t+2} = h_t h_{t+1}$. In the reputational equilibrium a competent type invests in all periods following t and in particular also in period $t+1$, if matched with a consumer. So, an investment at t generates a long-run price premium $p^{\text{ind}}(Gh_{t+1}) - p^{\text{ind}}(Bh_{t+1})$. If the firm is indeed matched with a consumer in period $t+1$ with a high enough probability (here this probability is fixed at $\frac{1}{2}$), then the firm might be tempted to rely on its future equilibrium investments, which would hurt its incentives to invest. However, in the case of exclusive knowledge for example, $p^{\text{ind}}(Gh_{t+1}) - p^{\text{ind}}(Bh_{t+1})$ is positive and potentially large if $h_{t+1} \in \{B, \varnothing\}$. The fact that the outcome in the next period may be $\varnothing$ may provide sufficient discipline for an individual brand to invest at t.

To completely characterize the threshold $\bar{c}^{\text{ind}}$, we need to calculate the minimum of $\bar{c}^{\text{ind}}(h_{t-1})$ over all outcomes $h_{t-1} \in \{G, B, \varnothing\}$. Figure 1 depicts $\bar{c}^{\text{ind}}(h_{t-1})$ as a function of the prior probability that a firm is competent, μ. As expected, the threshold vanishes at $\mu = 0$ and $\mu = 1$ because in these cases consumers' beliefs are unaffected by observed history, so the price premiums associated with an investment are zero. Obviously, in these cases, the firm cannot be induced to invest.

[Insert Figure 1 Here.]

Figure 1 also illustrates the history on which $\overline{c}^{\text{ind}}$ is attained. It shows that for a large μ, $\overline{c}^{\text{ind}} = \overline{c}^{\text{ind}}(G)$, that is, the firm faces the weakest incentive to invest after a good outcome. This is because observation of a good outcome pushes posterior beliefs further up, which tempts the firm to milk its good reputation. For a low value of μ, $\overline{c}^{\text{ind}} = \overline{c}^{\text{ind}}(B)$ because given a pessimistic prior, the observation of a bad outcome pushes posterior beliefs further down, which discourages the firm from investment because investment doesn't change beliefs sufficiently to justify it.

The next Lemma shows that this observation is true also more generally.

Lemma 1. *If μ is sufficiently large so that $\mu \geq \frac{\pi_L(1-\pi_L)}{\pi_H(1-\pi_H)+\pi_L(1-\pi_L)}$, then $\overline{c}^{ind} = \overline{c}^{ind}(G)$, and otherwise, $\overline{c}^{ind} = \overline{c}^{ind}(B)$.*

Collective Brand

If the two firms form a collective brand, then consumers observe the performance history of the collective brand $\mathbf{h}_t \in \{G, B\}^2$ without being able to distinguish whether it was produced by the particular firm with which they have been matched or the other firm in the collective brand. Compared to an individual brand, the history of a collective brand provides a noisier signal about firms' types, and instead a collective brand produces an outcome or signal in every period.

The general approach for the characterization of the reputational equilibrium under a collective brand is similar to the approach for an individual brand. We state the result for a collective brand in the following proposition.

Proposition 2. *A reputational equilibrium exists for a collective brand if and only if the cost of investment c is such that*

$$c \leq \bar{c}^{col} \quad \equiv \min_{h_{t-1}\in\{G,B\},\ \theta\in\{C,I\}} \bar{c}^{col}(h_{t-1},\theta)$$

where $\bar{c}^{col}(h_{t-1},\theta)$ denotes the expected benefit from an investment after history $\mathbf{h}_t = h_{t-2}h_{t-1}$ if the other firm is of type $\theta \in \{C, I\}$. It is given by

$$\begin{aligned} \bar{c}^{col}(h_{t-1},\theta) \equiv \ & \tfrac{\Delta\pi}{2}\cdot\delta\cdot\Big[\big(p^{col}(h_{t-1}G)-p^{col}(h_{t-1}B)\big)+ \\ & \delta\cdot \textstyle\sum_{h_{t+1}\in\{B,G\}}\pi(h_{t+1}|\theta)\cdot\left(p^{col}(Gh_{t+1})-p^{col}(Bh_{t+1})\right)\Big], \end{aligned} \tag{4}$$

where $\pi(h_{t+1}|\theta)$ denotes the probability distribution over the realized outcome in period $t+1$. If the other firm is competent, then $\pi(G|C) = \pi_H$ and $\pi(B|C) = 1 - \pi(G|C)$. If it is incompetent, then $\pi(G|I) = \frac{\pi_H+\pi_H}{2}$ and $\pi(B|I) = 1 - \pi(G|C)$.

As with an individual brand, the existence of a reputational equilibrium for a collective brand is characterized by a threshold rule. The difference between the two cases stems from the fact that in the case of a collective brand the investment incentives of a competent firm depend on the other firm's investment. Accordingly, the threshold cost $\bar{c}^{\text{col}}$ is the minimum of $\bar{c}^{\text{col}}(h_{t-1},\theta)$ over the history $h_{t-1} \in \{G, B\}$ and the other firm's type, $\theta \in \{C, I\}$, which is unobserved by consumers. As in the case of an individual brand, the function $\bar{c}^{\text{col}}(h_{t-1},\theta)$ can be expressed in terms of the primitives of the model, but because the resulting expression is long, we relegate it to Appendix A.

In the short-run, a competent firm in a collective brand expects a price premium of $p^{\text{col}}(h_{t-1}G) - p^{\text{col}}(h_{t-1}B)$ from investment which depends on the consumers' prior belief μ, and probabilities π_H and π_L. For example, in the case of exclusive knowledge, upon observation of an outcome $h_{t-1} = G$, the consumer learns that one firm in the collective is competent, but the type of the other firm remains unknown. This implies that the firm has an incentive to invest even after a good outcome in order to improve its reputation. In other words, unlike an individual brand, for a collective brand $p^{\text{col}}(h_{t-1}G) - p^{\text{col}}(h_{t-1}B) > 0$.[12] That is, consumers' limited information about individual firms within a collective brand mitigates each firm's short-run moral hazard problem.

In the long-run, the firm's investment in period t can contribute to its reputation in period $t+2$. The corresponding price premium generated by an investment is given by $p^{\text{col}}(Gh_{t+1}) - p^{\text{col}}(Bh_{t+1})$. A collective brand produces an outcome in every period, regardless of which firm consumer visits. So, a firm may free-ride on its own as well as on the other firm's future investment. This results in weaker long-run incentives to invest in a collective brand compared to an individual brand.

Hence, an individual brand faces a more severe commitment problem in the short-run, whereas a collective brand faces a bigger problem in the long-run. This tradeoff plays a central role in the comparison presented in the next subsection.

Figure 2 below depicts the expected return from investment $\bar{c}^{\text{col}}(h_{t-1}, \theta)$ for each history $h_{t-1} \in \{G, B\}$ and type $\theta \in \{C, I\}$ of the other firm. The solid line represents the threshold cost $\bar{c}^{\text{col}}$, which is given by the minimum of $\bar{c}^{\text{col}}(h_{t-1}, \theta)$ over $h_{t-1} \in \{G, B\}$ and $\theta \in \{C, I\}$, as a function of μ.

[Insert Figure 2 Here.]

If consumers' prior beliefs μ are very high, then a competent firm in a collective brand faces a commitment problem because it has a reputation that is good enough to exploit. This commitment problem is more severe after a good history $h_{t-1} = G$ when the firm expects the other firm to invest in the future ($\theta = C$). Thus, for high values of μ, $\bar{c}^{\text{col}} = \bar{c}^{\text{col}}(G, C)$.

If μ is small, then the firm is discouraged from investing further. This discouragement becomes more severe after a bad history $h_{t-1} = B$ if the firm expects the other firm to not invest in the future ($\theta = I$). Therefore, for low values of μ, $\bar{c}^{\text{col}} = \bar{c}^{\text{col}}(B, I)$. This insight is stated formally in the next Lemma.

Lemma 2. *For μ close to 1, $\bar{c}^{col} = \bar{c}^{col}(G, C)$, and for μ close to 0, $\bar{c}^{col} = \bar{c}^{col}(B, I)$.*

[12] Specifically, $p^{\text{col}}(GG) - p^{\text{col}}(GB) = \frac{3\mu+1}{2\mu+2} - \frac{1}{2} > 0$ and $p^{\text{col}}(BG) - p^{\text{col}}(BB) = \frac{1}{2} - \frac{\mu(1-\mu)}{2\mu^2-6\mu+4} > 0$.

Comparing Individual and Collective Brands

In this subsection, we examine the conditions under which a collective brand sustains the reputational equilibrium on a broader set of investment costs than an individual brand. When this is the case, a collective brand can be said to induce stronger incentives to invest than an individual brand. Other equilibria are discussed in Section 5. To highlight the critical mechanism, we focus on the case of exclusive knowledge where $\pi_L = 0$ and $\pi_H \in (0, 1)$. In this case, a good outcome G reveals competence, which allows us to derive results that are easy to interpret. An analogous result can be obtained for the case of quality control case as explained below.

The next proposition shows that in the case of exclusive knowledge, a collective brand sustains the reputational equilibrium for a broader set of costs than an individual brand if the discount factor δ is not too large. This is because, as explained previously, a collective brand provides stronger short-run incentives than individual reputation. If π_H is sufficiently large, then a collective brand induces stronger incentives to invest for every discount factor $\delta < 1$.

Proposition 3. *In the case of exclusive knowledge, there exists a threshold discount factor* $\overline{\delta} \in (0, 1]$ *such that* $\bar{c}^{col} > \bar{c}^{ind}$ *if and only if* $\delta < \overline{\delta}$. *If* π_H *is sufficiently close to* 1 *and* μ *is not too small, then* $\overline{\delta} = 1$. *In this case,* $\bar{c}^{col} > \bar{c}^{ind}$ *for all* $\delta \in [0, 1)$.

The fact that a collective brand induces stronger incentives to invest for all discount factors $\delta < 1$ if π_H is sufficiently large is because a higher π_H implies that signals are more accurate, which strengthens the incentive to milk reputation. This affects an individual brand more than a collective brand.

The magnitude of the short-run benefit provided by a collective brand depends critically on prior beliefs μ. In particular, in the case of exclusive knowledge, this benefit is greater if μ is large, or if consumers are optimistic about firms' competence. This is formalized in the following proposition.

Proposition 4. *In the case of exclusive knowledge, if δ is not too large, and if μ is sufficiently close to* 1*, then $\bar{c}^{col} > \bar{c}^{ind}$; if μ is close to* 0*, then $\bar{c}^{col} < \bar{c}^{ind}$.*

Intuitively, a larger μ increases the posterior probability that firms are competent, which as explained above, strengthens the firms' incentive to milk their reputation. This effect is stronger for an individual brand, because, in the case of a collective brand, the observation of a good outcome still allows for the possibility that the other firm in the collective is incompetent, which implies that the reputation of a collective brand is less sensitive to high prior beliefs μ.

If μ is small, then firms are concerned with building up a reputation. This takes place over time, which implies that firms' long-run incentives become more important. Consequently, individual reputation induces stronger incentives to invest.

[Insert Figure 3 Here.]

Figure 3 depicts the threshold costs for individual and collective reputation $\bar{c}^{\text{ind}}$ and $\bar{c}^{\text{col}}$, respectively. The higher is the threshold cost, the stronger is the incentive to invest. The figure shows that for μ close to 1, a collective brand dominates individual brands, although the opposite is true for μ close to 0. Proposition 3 implies that a collective brand is less attractive when δ is large, but the discount factor used in Figure 3 is $\delta = 0.9$, which shows that a collective brand dominates individual brands for a rather large set of parameter values.

In the case of quality control ($\pi_H = 1$ and $\pi_L \in (0, 1)$), it is possible to show that: (1) $\bar{c}^{\text{col}} > \bar{c}^{\text{ind}}$ if δ is not too large and μ is sufficiently close to 0, and (2) $\bar{c}^{\text{col}} < \bar{c}^{\text{ind}}$ if μ is sufficiently close to 1. In such an environment an individual brand's reputation plunges after a bad outcome, which discourages the firm from an investment. For a collective brand, a bad outcome provides noisier information because the other firm in the brand may still be competent. Consequently, a collective brand's reputation can still be recovered by additional investments.[13]

[13]See Appendix B for the formal statement and proof of this result.

Our observations seem to be consistent with observed practice. The parameter μ may be interpreted as the *baseline reputation* of firms in the market, industry, or country. Proposition 4 suggests that collective brands would likely thrive in exclusive knowledge industries (such as advanced electronics, automobiles, watches, etc.) in markets with a high baseline reputation, but less so in other markets. Indeed, car manufacturers in Germany often emphasized their country of origin.[14] In contrast, in the 1990s a Chinese consumer electronics and home appliances company Haier chose a foreign (more specifically, German) sounding name and tried to detach itself from its country of origin (see also Zhang, 2015).

5 Brand Formation

A natural question to ask is if a competent firm would ever want to form a brand with an incompetent firm. So far, we have examined each brand regime as exogenously given. This may be realistic in some applications. For example, a country might require each local manufacturer to label the goods it produces with a label of the country. Similarly, producers of wine, cheese, and coffee may become part of an appellation that is determined by their geographical location. However, in other applications, it is a firm's strategic decision whether to establish an individual brand or join a collective brand.

In this section, we illustrate that the benefit of a collective brand can be strong enough to provide a competent firm with an incentive to form a collective brand even with an incompetent firm.[15] We focus on the case that is described in Proposition 3 (where collective reputation induces stronger incentives for effort than individual reputation) where the cost of effort is such that it is possible to sustain the reputational equilibrium, which is the most profitable equilibrium, under collective reputation but not under individual reputation.

[14] Because of recent scandals, German car manufacturers currently have a poor collective reputation in terms of fraudulent emissions claims for their cars. It is not clear how quickly they will be able to restore their reputation.

[15] If a competent firm prefers to brand with an incompetent firm, then it obviously also prefers to brand with a competent firm. An incompetent firm prefers branding with a competent firm to forming an individual brand because being part of a collective brand ensures higher average prices.

We assume that firms make their branding decisions at the beginning of the game described in Section 3. After the types of the two firms are realized, the firms learn each other's type and decide whether to form individual brands or a collective brand. Then, firms and consumers play a stationary equilibrium of the game. We assume that a competent firm chooses the branding regime that induces an equilibrium that generates the highest profit.

Because we focus on the case where a collective brand can sustain the reputational equilibrium, to understand a competent firm's branding incentives, we need to identify the profit-maximizing equilibrium for an individual brand in the case we consider. For simplicity, in the discussion below, we confine our attention to stationary pure strategy equilibria (the proof of Proposition 5 allows for mixed strategy stationary equilibria).

Conveniently, as noted in Section 4, a competent firm's decision to invest depends only on the outcome produced in the previous period because by the next period the outcome produced two periods ago will be forgotten. Therefore, it follows that a stationary equilibrium strategy can be described by a subset $\mathcal{S}$ of the set of one-period histories, such that a competent firm invests after outcome h_t if and only if $h_t \in \mathcal{S}$. For example, if, say, $\mathcal{S} = \{B, \varnothing\}$, then a competent firm invests in period $t+1$ if and only if $h_t = B$ or $h_t = \varnothing$.

We calculate the expected per-period profits as follows. Conditional on the history $\mathbf{h}$, consumers facing an individual or collective brand $b \in \{\text{ind,col}\}$ update their beliefs that the firm is competent to $\Pr^b_{\mathcal{S}}(\mathbf{h})$. Then, the price the firm receives depends on the equilibrium strategy of the competent type, which is given by the probability that a competent firm invests after history $\mathbf{h}$, denoted $\sigma_{\mathcal{S}}(\mathbf{h}) \in \{0, 1\}$. As a result, the expected per-period profit of a competent firm conditional on the observed history $\mathbf{h}$ is given by:

$$\Pi^b_{\mathcal{S}}(\mathbf{h}) = \frac{1}{2}\left(\Pr^b_{\mathcal{S}}(\mathbf{h})\sigma_{\mathcal{S}}(\mathbf{h})\pi_H + \Pr^b_{\mathcal{S}}(\mathbf{h})(1-\sigma_{\mathcal{S}}(\mathbf{h}))\pi_L + (1-\Pr^{\mathrm{b}}_{\mathcal{S}}(\mathbf{h}))\pi_L - \sigma_{\mathcal{S}}(\mathbf{h})c\right),$$

which in the case of exclusive knowledge is equal to the equilibrium probability of producing high quality minus the cost. So, the expected per-period profit of firm i when firms' types are given by $\theta = (\theta_i, \theta_{-i}) \in \{C, I\}^2$, is given by:

$$\Pi^{\mathrm{b}}_{\theta,\mathcal{S}} \;=\; \sum_{\mathbf{h}} \pi^b_{\theta,\mathcal{S}}(\mathbf{h}) \cdot \Pi^b_{\mathcal{S}}(\mathbf{h}).$$

where $\pi^b_{\theta,\mathcal{S}}(\mathbf{h})$ denotes the stationary probability distribution over histories, which depends on θ and the equilibrium strategy $\sigma_{\mathcal{S}}$. For example, under the reputational equilibrium for a collective brand, i.e., $\mathcal{S} = \{G, B\}$,

$$\pi^{col}_{CC,\{G,B\}}(GG) = \pi_H^2, \quad \pi^{col}_{CI,\{G,B\}}(GG) = \pi^{col}_{IC,\{G,B\}}(GG) = \left(\tfrac{\pi_H+\pi_L}{2}\right)^2, \quad \pi^{col}_{II,\{G,B\}}(GG) = \pi_L^2$$

In the case of exclusive knowledge, if μ is sufficiently close to 1 and $c \in (\bar{c}^{\mathrm{ind}}, \bar{c}^{\mathrm{col}})$, then the unique equilibrium for an individual brand is the "no investment" equilibrium, which generates a zero profit. Therefore, a competent firm can attain a higher profit through the reputational equilibrium as a collective brand than as an individual brand. This benefit can be sufficiently large for a competent firm to want to form a collective brand even with an incompetent firm.

Proposition 5. *In the case of exclusive knowledge, suppose μ is sufficiently large and δ is not too large so that $\bar{c}^{ind} < \bar{c}^{col}$. Then, if $c \in (\bar{c}^{ind}, \bar{c}^{col})$, then a competent firm prefers forming a collective brand, even if it knows that the other firm is incompetent, to operating as an individual brand.*

Intuitively, the reason that in the case considered in Proposition 5, an individual brand can only sustain the no investment equilibrium is the following. An equilibrium where the competent firm invests only after the G or G and empty outcomes is difficult to sustain because per-period profits drop to zero after histories following which firms are not supposed to invest. So in equilibria where a competent firm is not supposed to invest after the outcome B, no investment only increases the likelihood that outcome B would repeat itself, which induces the firm to invest in preventing this situation. Moreover, an equilibrium where a competent firm invests only after the B or B and empty outcomes cannot be sustained because in such an equilibrium firms are not rewarded for the good outcome, because they are not expected to exert any effort after a good outcome. This implies that a competent firm has no incentive to exert efforts after the outcome B because doing so increases the probability that the next outcome would be G, and the firm's profit after outcome G is zero. This implies that a competent firm would benefit if it deviated and did not invest after the outcome B.

6 Extensions

T-period Memory

The intuition for why a collective brand may induce stronger incentives to invest does not depend on the length of consumers' memory. In this section, we extend consumers' memory to $T \geq 2$ periods and show that our main results are robust in the following sense: the range of discount factors δ for which a collective brand provides a stronger incentive to invest than an individual brand ($\bar{c}^{\text{ind}} < \bar{c}^{\text{col}}$) increases monotonically with T. Moreover, in the case of exclusive technology, the result holds for any discount factor as T tends to infinity.

In general, with a longer memory, each investment becomes less important. Thus, the benefit of a single investment decreases in T both for an individual and a collective brand. However, the benefit of investment is more adversely affected for individual brands. The intuition is identical to that for the 2-period memory. With a longer memory, an individual brand can reach more extreme reputations following a sequence of good or bad outcomes, which worsens the associated moral hazard problem.[16]

The following proposition generalizes Proposition 3 to T periods. A detailed analysis and proofs of the $T-$period case are relegated to Appendix B.

[16]For $T = 1$, an individual brand induces stronger incentives to invest than a collective brand because a firm cannot afford to rest on its laurels and must work continuously to maintain a good reputation. In this case, the more accurate signal that is produced by an individual brand induces stronger incentives for effort.

Proposition 6. *If* $\pi_L = 0$, $\pi_H \in (0,1)$, *and* μ *is sufficiently close to* 1, *then a collective brand sustains a reputational equilibrium for higher investment costs than an individual brand* ($\bar{c}^{ind} < \bar{c}^{col}$) *if the discount factor* δ *is sufficiently small. Moreover, the region of* δ *for which* $\bar{c}^{ind} < \bar{c}^{col}$ *increases monotonically (in the sense of set inclusion) in* T *and converges to* $[0,1]$ *as* T *tends to infinity.*

[Insert Figure 4 Here.]

Two panels in Figure 4 exhibits the range of parameters for which a collective brand induces stronger incentives to invest than an individual brand in the limit as μ tends to 1 and 0, respectively.[17] Panel (a) shows that for the case of exclusive knowledge, a larger δ requires a correspondingly longer memory T for a collective brand to outperform an individual brand. Panel (b) exhibits the analogous result for the case of quality control. Although the cutoff $\overline{\delta}$ is non-monotonic in T, we show in Appendix B that it converges to $\frac{1}{2}$.

More than Two Firms

In this subsection, we generalize the model by allowing for an arbitrary number of firms $n \geq 3$. We maintain the assumptions of a 2-period memory and that the consumer that arrives in each period is randomly matched with one firm. This implies that the sets of possible relevant histories are unchanged compared to the case where $n = 2$.

In the collective case, consumers facing a collective brand cannot distinguish between the identities of individual firms. They care about the expected quality of a randomly matched firm. Thus, the updating depends on the number of firms. In particular, the signal is weaker with more firms because the consumer knows that she is less likely to have observed the history of the firm with which she is matched. As is the case with $n = 2$, the reputational equilibrium exists for a collective brand if and only if the cost of investment is smaller than

[17] We calculated the limit threshold costs and present them in Appendix B. Because these threshold costs vanish at $\mu = 1$ and $\mu = 0$, we divide them by $1 - \mu$ and μ, respectively, and so compare $\frac{\bar{c}^{\text{col}}}{1-\mu}$ and $\frac{\bar{c}^{\text{ind}}}{1-\mu}$ in the case of exclusive knowledge and $\frac{\bar{c}^{\text{col}}}{\mu}$ and $\frac{\bar{c}^{\text{ind}}}{\mu}$ in the case of quality control.

or equal to a minimum threshold that describes the expected benefit from investment given histories and other firms' types.

Because free-riding incentives increase with the number of firms, the benefit of a collective reputation as a commitment device for investment decreases with n. However, as shown by the next proposition, it is still true that collective reputation induces stronger incentives for effort than individual reputation under conditions that are similar to those described in Proposition 3.

Proposition 7. *Suppose that $\pi_L = 0$ and $\pi_H \in (0,1)$. If consumers' prior belief about each firm's type μ is sufficiently close to 1, then a collective brand sustains a reputational equilibrium for higher investment costs than an individual brand ($\bar{c}^{col} > \bar{c}^{ind}$) if the discount factor δ is less than some threshold discount factor $\bar{\delta}_n < 1$. As μ approaches 1, $\bar{\delta}_n \to \frac{2(n-1)}{n(n^2-2)}$, which decreases in the number of firms, n.*

7 Conclusion

A reputation that is either too good or too bad discourages investment because of what may be interpreted as either complacency or despair, respectively. In this article, we show that a collective brand may induce stronger incentives to invest than an individual brand because it induces less extreme good and bad reputations.

Our analysis suggests that collective reputation may be preferable in "exclusive knowledge" industries in highly developed economies (such as French wine, Swiss watches, German automobiles, Japanese electronics, Italian fashion, US software, etc.), but less so in developing economies. On the other hand, collective reputation may also be preferable in "quality control" industries (such as screws, basic apparel, etc.) in developing economies. These theoretical results are consistent with anecdotal evidence about 'country-of-origin' labelling. For example, the collective brand "Made in China" is advertised by sub-suppliers on platforms such as Made-in-China.com, whereas successful high-tech companies such as Huawei try to build their brand names. On the other hand, Swiss watchmakers and Italian fashion houses emphasize their country of origin, although German sub-suppliers of generics, such as ThyssenKrupp, count on their brand reputation.

References

Andersson, F. "Pooling Reputations." *International Journal of Industrial Organization*, Vol. 20 (2002), pp. 715–730.

Bar-Isaac, H. "Something to Prove: Reputation in Teams." *The RAND Journal of Economics*, Vol. 38 (2007), pp. 495–511.

Cabral, L. M. B. "Stretching Firm and Brand Reputation." *The RAND Journal of Economics*, Vol. 31 (2000), pp. 658–673.

Choi, J. P. "Brand Extension as Informational Leverage." *The Review of Economic Studies*, Vol. 65 (1998), pp. 655–669.

Fedele, A., and Tedeschi, P. "Reputation and Competition in a Hidden Action Model." *PloS one*, Vol. 9 (2014).

Fishman, A., Finkelstein, I., Simhon, A., and Yacouel, N. "Collective Brands." *International Journal of Industrial Organization*, Vol. 59 (2018), pp. 316–339.

Fleckinger, P. (2016) "Regulating Collective Reputation Markets." Working paper.

Fleckinger, P., Mimra, W., and Zago, A. (2017) "The Incentive Properties of Collective Reputation." Working paper.

Hakenes, H., and Peitz, M. "Umbrella Branding and the Provision of Quality." *International Journal of Industrial Organization*, Vol. 26 (2008), pp. 546–556.

Holmström, B. "Managerial Incentive Problems: A Dynamic Perspective." *The Review of Economic Studies*, Vol. 66 (1999), pp. 169–182.

Hörner, J. "Reputation and Competition." *American Economic Review*, Vol. 92 (2002), pp. 644–663.

Klein, B., and Leffler, K. B. "The Role of Market Forces in Assuring Contractual Performance." *Journal of Political Economy*, Vol. 89 (1981), pp. 615–641.

Kranton, R. E. "Competition and the incentive to produce high quality." *Economica*, Vol. 70 (2003), pp. 385–404.

Levin, J. “The Dynamics of Collective Reputation.” *The BE Journal of Theoretical Economics*, Vol. 9 (2009).

Liu, Q., and Skrzypacz, A. “Reputation and Competition.” *Journal of Economic Theory*, Vol. 151 (2014), pp. 2–29.

Mailath, G. J., and Samuelson, L. “Who Wants a Good Reputation?.” *The Review of Economic Studies*, Vol. 68 (2001), pp. 415–441.

Miklós-Thal, J. “Linking Reputations through Umbrella Branding.” *Quantitative Marketing and Economics*, Vol. 10 (2012), pp. 335–374.

Moav, O., and Neeman, Z. “The Quality of Information and Incentives for Effort.” *The Journal of Industrial Economics*, Vol. 58 (2010), pp. 642–660.

Moorthy, S. “Can Brand Extension Signal Product Quality?.” *Marketing science*, Vol. 31 (2012), pp. 756–770.

Nelson, P. “Information and consumer behavior.” *Journal of Political Economy*, Vol. 78 (1970), pp. 311–329.

Nosko, C., and Tadelis, S. (2015) “The limits of reputation in platform markets: An empirical analysis and field experiment.” Working paper.

Tirole, J. “A Theory of Collective Reputations (with Applications to the Persistence of Corruption and to Firm Quality).” *The Review of Economic Studies*, Vol. 63 (1996), pp. 1–22.

Wernerfelt, B. “Umbrella Branding as a Signal of New Product Quality: An Example of Signalling by Posting a Bond.” Vol. (1988), pp. 458–466.

Winfree, J. A., and McCluskey, J. J. “Collective reputation and quality.” *American Journal of Agricultural Economics*, Vol. 87 (2005), pp. 206–213.

Yu, J. (2018) “A Model of Brand Architecture Choice: A Branded House vs. A House of Brands.” Working paper.

Zhang, K. “Breaking free of a stereotype: Should a domestic brand pretend to be a foreign one?.” *Marketing Science*, Vol. 34 (2015), pp. 539–554.

A Appendix: Proofs

Proof. [Proposition 1] The posterior beliefs $\Pr^{\text{ind}}$ about the quality of the product after observing history $\mathbf{h}_t = h_{t-2}h_{t-1}$ are given by Bayes' rule:

$$\Pr^{\text{ind}}(GG) = \frac{\mu\pi_H^2}{\mu\pi_H^2 + (1-\mu)\pi_L^2}, \qquad \Pr^{\text{ind}}(GB) = \Pr^{\text{ind}}(BG) = \frac{\mu\pi_H(1-\pi_H)}{\mu\pi_H(1-\pi_H) + (1-\mu)\pi_L(1-\pi_L)},$$
$$\Pr^{\text{ind}}(BB) = \frac{\mu(1-\pi_H)^2}{\mu(1-\pi_H)^2 + (1-\mu)(1-\pi_L)^2}, \qquad \Pr^{\text{ind}}(G\varnothing) = \Pr(\varnothing G) = \frac{\mu\pi_H}{\mu\pi_H + (1-\mu)\pi_L}$$
$$\Pr^{\text{ind}}(\varnothing\varnothing) = \mu, \qquad \Pr^{\text{ind}}(B\varnothing) = \Pr^{\text{ind}}(\varnothing B) = \frac{\mu(1-\pi_H)}{\mu(1-\pi_H) + (1-\mu)(1-\pi_L)}.$$

The reputational equilibrium exists if and only if a competent firm invests whenever visited following all histories, i.e.,

$$p^{\text{ind}}(h_{t-2}h_{t-1}) - c + \delta \cdot \big(\pi_H V(h_{t-1}G) + (1-\pi_H)V(h_{t-1}B)\big) \geq$$
$$p^{\text{ind}}(h_{t-2}h_{t-1}) + \delta \cdot \big(\pi_L V(h_{t-1}G) + (1-\pi_L)V(h_{t-1}B)\big)$$

which is equivalent to

$$c \leq \overline{c}^{\text{ind}}(h_{t-1}) := \delta \cdot (\pi_H - \pi_L) \cdot (V(h_{t-1}G) - V(h_{t-1}B)).$$

Then, $V(h_{t-1}G)$ and $V(h_{t-1}B)$ can be written as

$$V(h_{t-1}G) = \frac{p^{\text{ind}}(h_{t-1}G) - c}{2} + \frac{\delta}{2} \cdot \big(\pi_H V(GG) + (1-\pi_H)V(GB) + V(G\varnothing)\big)$$
$$V(h_{t-1}B) = \frac{p^{\text{ind}}(h_{t-1}B) - c}{2} + \frac{\delta}{2} \cdot \big(\pi_H V(BG) + (1-\pi_H)V(BB) + V(B\varnothing)\big).$$

Then, the difference is

$$V(h_{t-1}G) - V(h_{t-1}B) = \frac{p^{\text{ind}}(h_{t-1}G) - p^{\text{ind}}(h_{t-1}B)}{2} + \frac{\delta}{2} \cdot \Big(\pi_H \underbrace{(V(GG) - V(BG))}_{\frac{p^{\text{ind}}(GG) - p^{\text{ind}}(GB)}{2}}$$
$$+ (1-\pi_H)\underbrace{(V(GB) - V(BB))}_{\frac{p^{\text{ind}}(GB) - p^{\text{ind}}(BB)}{2}} + \underbrace{V(G\varnothing) - V(B\varnothing)}_{\frac{p^{\text{ind}}(G\varnothing) - p^{\text{ind}}(B\varnothing)}{2}}\Big).$$

□

Proof. [Lemma 1] From Equation (3), a comparison across $\overline{c}^{\text{ind}}(h_{t-1})$ for $h_{t-1} \in \{G, B, \varnothing\}$

hinges on the comparison across per-period payoffs, i.e., $p^{\text{ind}}(h_{t-1}G) - p^{\text{ind}}(h_{t-1}B)$

$$\begin{aligned}
p^{\text{ind}}(GG) - p^{\text{ind}}(GB) &= (\pi_H - \pi_L) \cdot \left(\frac{\mu \pi_H^2}{\mu \pi_H^2 + (1-\mu)\pi_L^2} - \frac{\mu \pi_H (1-\pi_H)}{\mu \pi_H (1-\pi_H) + (1-\mu)\pi_L(1-\pi_L)} \right) \\
&= \frac{\mu(1-\mu)\pi_H \pi_L (\pi_H - \pi_L)^2}{\big(\mu \pi_H^2 + (1-\mu)\pi_L^2\big) \cdot \big(\mu \pi_H (1-\pi_H) + (1-\mu)\pi_L(1-\pi_L)\big)}, \\
p^{\text{ind}}(GB) - p^{\text{ind}}(BB) &= (\pi_H - \pi_L) \cdot \left(\frac{\mu \pi_H (1-\pi_H)}{\mu \pi_H (1-\pi_H) + (1-\mu)\pi_L(1-\pi_L)} - \frac{\mu (1-\pi_H)^2}{\mu (1-\pi_H)^2 + (1-\mu)(1-\pi_L)^2} \right) \\
&= \frac{\mu(1-\mu)(1-\pi_H)(1-\pi_L)(\pi_H - \pi_L)^2}{\big(\mu \pi_H (1-\pi_H) + (1-\mu)\pi_L(1-\pi_L)\big) \cdot \big(\mu (1-\pi_H)^2 + (1-\mu)(1-\pi_L)^2\big)}.
\end{aligned}$$

And, for $h_{t-1} = \varnothing$,

$$\begin{aligned}
p^{\text{ind}}(G\varnothing) - p^{\text{ind}}(B\varnothing) &= (\pi_H - \pi_L) \cdot \left(\frac{\mu \pi_H}{\mu \pi_H + (1-\mu)\pi_L} - \frac{\mu (1-\pi_H)}{\mu (1-\pi_H) + (1-\mu)(1-\pi_L)} \right) \\
&= \frac{\mu(1-\mu)(\pi_H - \pi_L)^2}{\big(\mu \pi_H + (1-\mu)\pi_L\big) \cdot \big(\mu (1-\pi_H) + (1-\mu)(1-\pi_L)\big)}
\end{aligned}$$

$\overline{c}^{\text{ind}} = \overline{c}^{\text{ind}}(G)$ if and only if $p^{\text{ind}}(GG) - p^{\text{ind}}(GB) < p^{\text{ind}}(BG) - p^{\text{ind}}(BB)$ and $p^{\text{ind}}(GG) - p^{\text{ind}}(GB) < p^{\text{ind}}(\varnothing G) - p^{\text{ind}}(\varnothing B)$.

First, $p^{\text{ind}}(GG) - p^{\text{ind}}(GB) < p^{\text{ind}}(BG) - p^{\text{ind}}(BB)$ if and only if

$$\begin{aligned}
& \frac{\pi_H \pi_L}{\big(\mu \pi_H^2 + (1-\mu)\pi_L^2\big)} < \frac{(1-\pi_H)(1-\pi_L)}{\big(\mu (1-\pi_H)^2 + (1-\mu)(1-\pi_L)^2\big)} \\
\Leftrightarrow \quad & \pi_H \pi_L \big(\mu (1-\pi_H)^2 + (1-\mu)(1-\pi_L)^2\big) < (1-\pi_H)(1-\pi_L)(\mu \pi_H^2 + (1-\mu)\pi_L^2) \\
\Leftrightarrow \quad & (1-\mu)\pi_L(1-\pi_L)(\pi_H - \pi_L) < \mu \pi_H (1-\pi_H)(\pi_H - \pi_L) \\
\Leftrightarrow \quad & \mu > \bar{\mu} := \frac{\pi_L(1-\pi_L)}{\pi_H(1-\pi_H) + \pi_L(1-\pi_L)}
\end{aligned} \tag{5}$$

Second, $p^{\text{ind}}(GG) - p^{\text{ind}}(GB) < p^{\text{ind}}(\varnothing G) - p^{\text{ind}}(\varnothing B)$ if and only if

$$\frac{\pi_H \pi_L}{\big(\mu \pi_H^2 + (1-\mu)\pi_L^2\big) \cdot \big(\mu \pi_H (1-\pi_H) + (1-\mu)\pi_L(1-\pi_L)\big)} < \frac{1}{\big(\mu \pi_H + (1-\mu)\pi_L\big) \cdot \big(\mu (1-\pi_H) + (1-\mu)(1-\pi_L)\big)}.$$

This holds if $\mu > \hat{\mu}_1 := \frac{\pi_L^2(1-\pi_L)}{\pi_L^2(1-\pi_L) - \pi_H^2(1-\pi_H)} + \sqrt{\frac{\pi_H^2 \pi_L^2 (1-\pi_H)(1-\pi_L)}{\pi_L^2(1-\pi_L) - \pi_H^2(1-\pi_H)}}$. And, it can be shown that $\overline{\mu} > \hat{\mu}_1$.

Therefore, $\overline{c}^{\text{ind}} = \overline{c}^{\text{ind}}(G)$ if and only if $p^{\text{ind}}(GG) - p^{\text{ind}}(GB) < \min\{p^{\text{ind}}(BG) - p^{\text{ind}}(BB),\, p^{\text{ind}}(\varnothing, G) - p^{\text{ind}}(\varnothing, B)\}$, which holds if $\mu > \overline{\mu}$.

Otherwise, $p^{\text{ind}}(BG) - p^{\text{ind}}(BB) < p^{\text{ind}}(\varnothing G) - p^{\text{ind}}(\varnothing B)$ if and only if $\mu < \hat{\mu}_2 := \frac{\pi_L(1-\pi_L)^2}{\pi_L(1-\pi_L)^2 + \pi_H(1-\pi_H)^2} + \sqrt{\frac{(1-\pi_H)^2 \pi_H (1-\pi_L)^2 \pi_L}{\pi_L(1-\pi_L)^2 + \pi_H(1-\pi_H)^2}}$. It can be shown that $\overline{\mu} < \hat{\mu}_2$. Therefore, $\overline{c}^{\text{ind}} = \overline{c}^{\text{ind}}(B)$ if and only if $\mu < \overline{\mu}$.

Moreover, $\overline{\mu} \to 0$ as $\pi_L \to 0$. Therefore, for $\pi_L = 0$, $\overline{c}^{\text{ind}} = \overline{c}^{\text{ind}}(G)$ for all parameter regions. Also, for $\pi_H = 1$, $\overline{\mu} = 1$, and therefore $\overline{c}^{\text{ind}} = \overline{c}^{\text{ind}}(B)$ for all parameter regions.

□

Proof. [Proposition 2] Let us denote by $V(h;\theta)$ the present discounted expected equilibrium profit of a competent firm when branding with a θ-type firm after history $\mathbf{h}_t \in \mathcal{H}^{\text{col}}$ at the beginning of the period before the consumer is assigned to either firm.

Then, a reputational equilibrium exists if and only if for all $\mathbf{h}_t$, and θ, the following holds:

$$p^{\text{col}}(h_{t-2}h_{t-1}) - c + \delta \cdot \big(\pi_H V(h_{t-1}G;\theta) + (1-\pi_H)V(h_{t-1}B;\theta)\big) \geq \\ p^{\text{col}}(h_{t-2}h_{t-1}) + \delta \cdot \big(\pi_L V(h_{t-1}G;\theta) + (1-\pi_L)V(h_{t-1}B;\theta)\big).$$

This is equivalent to

$$c \leq \overline{c}^{\text{col}}(h_{t-1}) \equiv \delta \cdot (\pi_H - \pi_L) \cdot (V(h_{t-1}G;\theta) - V(h_{t-1}B;\theta)).$$

First, note that for all $q_1, q_2, x \in \{G, B\}$, we have that $V(q_1x,\theta) - V(q_2x,\theta) = \frac{p^{\text{col}}(q_1x) - p^{\text{col}}(q_2x)}{2}$. Using this, we can calculate

$$\begin{aligned} V(h_{t-1}G;\theta) - V(h_{t-1}B;\theta) &= \frac{p^{\text{col}}(h_{t-1}G) - p^{\text{col}}(h_{t-1}B)}{2} \\ &+ \frac{\delta\pi_H}{2} \underbrace{(V(GG,\theta) - V(BG,\theta))}_{\frac{p^{\text{col}}(GG)-p^{\text{col}}(BG)}{2}} + \frac{\delta(1-\pi_H)}{2} \underbrace{(V(GB,\theta) - V(BB,\theta))}_{\frac{p^{\text{col}}(GB)-p^{\text{col}}(BB)}{2}} \\ &+ \frac{\delta\pi(\theta)}{2}(V(GG,\theta) - V(BG,\theta)) + \frac{\delta(1-\pi(\theta))}{2}(V(GB,\theta) - V(BB,\theta)) \end{aligned}$$

where $\pi(\theta) = \pi_L$ if $\theta = I$ and π_H if $\theta = C$. □

Proof. [Lemma 2] As noted in Section 3, upon observing a history $\mathbf{h}_t \in \mathcal{H}^{\text{col}}$, a consumer places a probability $\eta_s(\mathbf{h}_t)$ on the group's type $s \in \{CC, CI, IC, II\}$. These beliefs are given by:

$$\begin{aligned}
\eta_{CC}(GG) &= \frac{\mu^2\pi_H^2}{\mu^2\pi_H^2 + 2\mu(1-\mu)\left(\frac{1}{4}\pi_H^2 + \frac{1}{2}\pi_H\pi_L + \frac{1}{4}\pi_L^2\right) + (1-\mu)^2\pi_L^2}, \\
\eta_{CI}(GG) &= \eta_{IC}(GG) \\
&= \frac{\mu(1-\mu)\left(\frac{1}{4}\pi_H^2 + \frac{1}{2}\pi_H\pi_L + \frac{1}{4}\pi_L^2\right)}{\mu^2\pi_H^2 + 2\mu(1-\mu)\left(\frac{1}{4}\pi_H^2 + \frac{1}{2}\pi_H\pi_L + \frac{1}{4}\pi_L^2\right) + (1-\mu)^2\pi_L^2}, \\
\eta_{II}(GG) &= 1 - \eta_{CC}(GG) - 2\eta_{CI}(GG), \\
\eta_{CC}(GB) &= \frac{\mu^2\pi_H(1-\pi_H)}{\mu^2\pi_H(1-\pi_H) + 2\mu(1-\mu)\frac{1}{4}\left(\pi_H(1-\pi_H) + \pi_H(1-\pi_L) + \pi_L(1-\pi_H) + \pi_L(1-\pi_L)\right) + (1-\mu)^2\pi_L(1-\pi_L)}, \\
\eta_{CI}(GB) &= \eta_{IC}(GB) \\
&= \frac{\mu(1-\mu)\frac{1}{4}\left(\pi_H(1-\pi_H) + \pi_H(1-\pi_L) + \pi_L(1-\pi_H) + \pi_L(1-\pi_L)\right)}{\mu^2\pi_H(1-\pi_H) + 2\mu(1-\mu)\frac{1}{4}\left(\pi_H(1-\pi_H) + \pi_H(1-\pi_L) + \pi_L(1-\pi_H) + \pi_L(1-\pi_L)\right) + (1-\mu)^2\pi_L(1-\pi_L)}, \\
\eta_{II}(GB) &= 1 - \eta_{CC}(GB) - 2\eta_{CI}(GB), \\
\eta_{CC}(BB) &= \frac{\mu^2(1-\pi_H)^2}{\mu^2(1-\pi_H)^2 + 2\mu(1-\mu)\left(\frac{1}{4}(1-\pi_H)^2 + \frac{1}{2}(1-\pi_H)(1-\pi_L) + \frac{1}{4}(1-\pi_L)^2\right) + (1-\mu)^2(1-\pi_L)^2}, \\
\eta_{CI}(BB) &= \eta_{IC}(BB) \\
&= \frac{\mu(1-\mu)\left(\frac{1}{4}(1-\pi_H)^2 + \frac{1}{2}(1-\pi_H)(1-\pi_L) + \frac{1}{4}(1-\pi_L)^2\right)}{\mu^2(1-\pi_H)^2 + 2\mu(1-\mu)\left(\frac{1}{4}(1-\pi_H)^2 + \frac{1}{2}(1-\pi_H)(1-\pi_L) + \frac{1}{4}(1-\pi_L)^2\right) + (1-\mu)^2(1-\pi_L)^2}, \\
\eta_{II}(BB) &= 1 - \eta_{CC}(BB) - 2\eta_{CI}(BB).
\end{aligned}$$

Then, the consumer's posterior belief is about the firm being competent is given by

$$\Pr(\mathbf{h}_t) = \eta_{CC}(\mathbf{h}_t) + \frac{1}{2}(\eta_{CI}(\mathbf{h}_t) + \eta_{IC}(\mathbf{h}_t))$$

and $p^{\text{col}}(\mathbf{h}_t) = \Pr(\mathbf{h}_t) \cdot \pi_H + (1 - \Pr(\mathbf{h}_t)) \cdot \pi_L$. Thus, the differences in prices are given by:

$$\begin{aligned}
p^{\text{col}}(GG) - p^{\text{col}}(GB) &= \frac{\mu(1-\mu)(\pi_H-\pi_L)^2\left(\mu^2(\pi_H-\pi_L)^2 + 2\mu(\pi_H-\pi_L)\pi_L + \pi_L(\pi_H+\pi_L)\right)}{4\cdot\Pr(GG)\cdot\Pr(GB)} \\
p^{\text{col}}(GB) - p^{\text{col}}(BB) &= \frac{\mu(1-\mu)(\pi_H-\pi_L)^2\left(\mu^2(\pi_H-\pi_L)^2 - 2\mu(\pi_H-\pi_L)(1-\pi_L) + (1-\pi_L)(2-\pi_H-\pi_L)\right)}{4\cdot\Pr(GB)\cdot\Pr(BB)}.
\end{aligned}$$

Thus, $p^{\text{col}}(GG) - p^{\text{col}}(GB) < p^{\text{col}}(GB) - p^{\text{col}}(BB)$ if and only if

$$\frac{\mu^2(\pi_H-\pi_L)^2 + 2\mu(\pi_H-\pi_L)\pi_L + \pi_L(\pi_H+\pi_L)}{\Pr(GG)} < \frac{\mu^2(\pi_H-\pi_L)^2 - 2\mu(\pi_H-\pi_L)(1-\pi_L) + (1-\pi_L)(2-\pi_H-\pi_L)}{\Pr(BB)}$$

Taking the limit $\mu \to 1$ on both sides, the inequality becomes

$$\frac{\pi_H(\pi_H+\pi_L)}{\pi_H^2} < \frac{(\pi_H-\pi_L)^2 - 2(\pi_H-\pi_L)(1-\pi_L) + (1-\pi_L)(2-\pi_H-\pi_L)}{(1-\pi_H)^2}.$$

This is equivalent to $\pi_L(1-\pi_H) < \pi_H(1-\pi_H)$, i.e., it is always satisfied. Similarly, for $\mu \to 0$, the inequality is equivalent to

$$\frac{\pi_L(\pi_H+\pi_L)}{\pi_L^2} < \frac{(1-\pi_L)(2-\pi_H-\pi_L)}{(1-\pi_L)^2}$$

which simplifies to $\pi_H < \pi_L$ which is never satisfied. Thus, by continuity $p^{\text{col}}(GG) - p^{\text{col}}(GB) < p^{\text{col}}(GB) - p^{\text{col}}(BB)$ for sufficiently large μ and $p^{\text{col}}(GG) - p^{\text{col}}(GB) > p^{\text{col}}(GB) - p^{\text{col}}(BB)$ for sufficiently small μ. The statement of the proposition follows from the definition of $\bar{c}^{\text{col}}(h_{t-1}, \theta)$ in (4).

One can show that unlike in the independent branding case, as μ increases, the binding history changes from B to G, back to B and then to G, but it does not yield additional insights, so we omit the proof and statement. □

Proof. [Proposition 3] It follows from Proposition 1 that if $\pi_L = 0$ for μ sufficiently large $\bar{c}^{\text{ind}}(G)$ determines the cutoff cost. Also, by Proposition 2, for sufficiently large μ, $\bar{c}^{\text{col}}(G; C)$ determines $\bar{c}^{\text{col}}$. Thus, it suffices to compare $\bar{c}^{\text{ind}} = \bar{c}^{\text{ind}}(G)$ and $\bar{c}^{\text{col}} = \bar{c}^{\text{col}}(G; C)$.

First, for an individual brand,

$$\begin{aligned}
\bar{c}^{\text{ind}}(G) &= \delta \cdot \frac{\pi_H}{2} \Bigg(\big(1 + \frac{\delta \pi_H}{2}\big) \big(\underbrace{p^{\text{ind}}(GG) - p^{\text{ind}}(GB)}_{=0} \big) \\
&\quad + \frac{\delta(1-\pi_H)}{2} \big(p^{\text{ind}}(GB) - p^{\text{ind}}(BB) \big) + \frac{\delta}{2} (p^{\text{ind}}(G\varnothing) - p^{\text{ind}}(B\varnothing)) \Bigg) \\
&= \delta^2 \cdot \frac{\pi_H^2}{2} (1-\mu) \left(\frac{1-\pi_H}{2} \cdot \frac{1}{1 - \mu\pi_H(2-\pi_H)} + \frac{1}{2} \cdot \frac{1}{1-\mu\pi_H} \right). \qquad (6)
\end{aligned}$$

Then, $\lim_{\mu \to 1} \frac{1}{1-\mu} \bar{c}^{\text{ind}}(G) = \frac{\delta^2 \pi_H^2}{2(1-\pi_H)}$.

For a collective brand,

$$\begin{aligned}
\bar{c}^{\text{col}}(G; C) &= \delta \cdot \frac{\pi_H}{2} \Big((1 + \delta\pi_H)(p^{\text{col}}(GG) - p^{\text{col}}(GB)) + \delta(1-\pi_H)(p^{\text{col}}(GB) - p^{\text{col}}(BB)) \Big) \\
&= \delta \cdot \frac{\pi_H}{2} (1 + \delta\pi_H) \cdot \frac{(1-\mu)\mu\pi_H}{(1+\mu)(2-(1+\mu)\pi_H)} \\
&\quad + \delta^2 \cdot \frac{\pi_H}{2} \cdot (1-\pi_H) \cdot \frac{(1-\mu)\pi_H \left(2 - \pi_H(1 + \mu(2-\mu\pi_H))\right)}{\left((1-\mu\pi_H)^2 + \mu(1-\pi_H)^2 + 1 - \mu\right)(2-(1+\mu)\pi_H)} \qquad (7)
\end{aligned}$$

Thus, $\lim_{\mu \to 1} \frac{1}{1-\mu} \bar{c}^{\text{col}}(G; C) = \frac{\delta}{2} \frac{\pi_H^2(1+2\delta)}{4(1-\pi_H)}$.

So, $\lim_{\mu \to 1} \frac{1}{1-\mu} \bar{c}^{\text{col}}(G; C) > \lim_{\mu \to 1} \frac{1}{1-\mu} \bar{c}^{\text{ind}}(G)$ if and only if $\delta < \frac{1}{2}$. Thus, for π_L sufficiently close to 0 and μ sufficiently close to 1, $\bar{c}^{\text{col}} \geq \bar{c}^{\text{ind}}$ if δ is not too large.

Moreover, if $\pi_L = 0$, and π_H sufficiently close to 1, then $\lim_{\pi_H \to 1} \bar{c}^{\text{col}}(G; C) > \lim_{\pi_H \to 1} \bar{c}^{\text{ind}}(G)$ if and only if $\frac{\delta(1+\delta)}{2} \cdot \frac{\mu}{1+\mu} > \frac{\delta^2}{4}$, which holds true if and only if $\delta < \frac{2\mu}{1+\mu}$. This condition holds for all $\delta \in (0, 1)$ if $\mu > \frac{1}{3}$. □

Proof. [Proposition 4] The statement for a μ close to 1 is proven in Proposition 3. For the case of a μ close to 0, plugging in $\mu = 0$ results in the following:

$$\begin{aligned} \lim_{\mu \to 0} \bar{c}^{\text{ind}}(G) &= \delta^2 \cdot \frac{2-\pi_H}{2} \cdot \frac{\pi_H^2}{2}, \\ \lim_{\mu \to 0} \bar{c}^{\text{col}}(G) &= \delta^2 \cdot \frac{\pi_H(1-\pi_H)}{2} \cdot \frac{\pi_H(2-\pi_H)}{2(2-\pi_H)}. \end{aligned}$$

Therefore, $\lim_{\mu \to 0} \bar{c}^{\text{ind}}(G) \geq \lim_{\mu \to 0} \bar{c}^{\text{col}}(G)$ for all parameter values $\delta \in (0,1)$ and $\pi_H \in (0,1)$. So, if $\pi_L = 0$, $\bar{c}^{\text{ind}} \geq \bar{c}^{\text{col}}$ for μ sufficiently close to 0. □

Proof. [Proposition 5] In the following we fully characterize the profit maximizing equilibria for the exclusive knowledge case $\pi_L = 0$, $\pi_H \in (0,1)$. We prove Proposition 5 in two steps. First, we characterize the set of equilibria for each branding regime b $\in$ {ind, col} and any cost $c > 0$. Then, we calculate the mean expected per-period profit fo a competent firm.

I. Individual brand: As discussed in the main text, every equilibrium can be characterized by a subset $\mathcal{S} \subset \{G, B, \varnothing\}$ and mixing probabilities on $\{G, B, \varnothing\} \setminus \mathcal{S}$. For each subset $\mathcal{S} \subset \{G, B, \varnothing\}$, we focus first on the pure strategy equilibrium and then extend the argument to a situation when the firm mixes after states in the complement.

For all equilibria other than the reputational equilibrium and no investment equilibrium, the competent type invests in quality after some histories, whereas it does not invest after other histories. Thus, such an equilibrium can only be sustained if the cost of investment is not too large or too small. In other words, for a pure-strategy equilibrium specified by a subset $\mathcal{S} \subset \{G, B, \varnothing\}$, there exist $\overline{C}_{\mathcal{S}}^{\text{ind}}$ and $\underline{C}_{\mathcal{S}}^{\text{ind}}$ such that the equilibrium exists if and only if $c \in [\underline{C}_{\mathcal{S}}^{\text{ind}}, \overline{C}_{\mathcal{S}}^{\text{ind}}]$. Formally, for any $s \in \mathcal{S}$, the following inequality must hold

$$c \leq \delta \cdot (\pi_H - \pi_L)(V_{\mathcal{S}}^{\text{ind}}(sG) - V_{\mathcal{S}}^{\text{ind}}(sB)) \tag{8}$$

where $V_{\mathcal{S}}^{\text{ind}}(h)$ is the equilibrium continuation payoff after a history h in equilibrium $\mathcal{S}$ at the beginning of the period before the consumer is assigned to either firm. The reverse inequality must hold for all $s \notin \mathcal{S}$. Note that for mixed equilibria the inequality must hold with equality for histories after which the firm mixes between investment and no investment.

For all $s \in \mathcal{S}$, we can write

$$V_{\mathcal{S}}^{\text{ind}}(h_{-2}s) \quad = \quad \frac{1}{2} \cdot \left(p_{\mathcal{S}}^{\text{ind}}(h_{-2}s) - c + \delta(\pi_H V_{\mathcal{S}}^{\text{ind}}(sG) + (1-\pi_H)V_{\mathcal{S}}^{\text{ind}}(sB) + V_{\mathcal{S}}^{\text{ind}}(s\varnothing))\right)$$

and for all $s \notin \mathcal{S}$

$$V_{\mathcal{S}}^{\text{ind}}(h_{-2}s) = \frac{1}{2} \cdot \left(p_{\mathcal{S}}^{\text{ind}}(h_{-2}s) + \delta(\pi_L V_{\mathcal{S}}^{\text{ind}}(sG) + (1-\pi_L)V_{\mathcal{S}}^{\text{ind}}(sB) + V_{\mathcal{S}}^{\text{ind}}(s\varnothing))\right)$$

I–1. Existence of $\{G,\varnothing\}-$ and $\{G\}-$equilibria.

Let $X := V_{\mathcal{S}}^{\text{ind}}(GG) - V_{\mathcal{S}}^{\text{ind}}(GB)$ and $Y := V_{\mathcal{S}}^{\text{ind}}(BG) - V_{\mathcal{S}}^{\text{ind}}(BB)$. Then, we can write

$$\begin{aligned} X &= \frac{p_{\mathcal{S}}^{\text{ind}}(GG) - p_{\mathcal{S}}^{\text{ind}}(GB) - c}{2} + \frac{\delta}{2} \cdot \Bigg(\pi_H \underbrace{(V_{\mathcal{S}}^{\text{ind}}(GG) - V_{\mathcal{S}}^{\text{ind}}(GB))}_{=X} - \pi_L \underbrace{(V_{\mathcal{S}}^{\text{ind}}(BG) - V_{\mathcal{S}}^{\text{ind}}(BB))}_{=Y} \\ &\quad + \underbrace{V_{\mathcal{S}}^{\text{ind}}(GB) - V_{\mathcal{S}}^{\text{ind}}(BB)}_{=0} + \underbrace{V_{\mathcal{S}}^{\text{ind}}(G\varnothing) - V_{\mathcal{S}}^{\text{ind}}(B\varnothing)}_{=\frac{p_{\mathcal{S}}^{\text{ind}}(G\varnothing) - p_{\mathcal{S}}^{\text{ind}}(B\varnothing)}{2}}\Bigg) \\ Y &= \frac{p_{\mathcal{S}}^{\text{ind}}(BG) - p_{\mathcal{S}}^{\text{ind}}(BB) - c}{2} + \frac{\delta}{2} \cdot \Bigg(\pi_H \underbrace{(V_{\mathcal{S}}^{\text{ind}}(GG) - V_{\mathcal{S}}^{\text{ind}}(GB))}_{=X} - \pi_L \underbrace{(V_{\mathcal{S}}^{\text{ind}}(BG) - V_{\mathcal{S}}^{\text{ind}}(BB))}_{=Y} \\ &\quad + \underbrace{V_{\mathcal{S}}^{\text{ind}}(GB) - V_{\mathcal{S}}^{\text{ind}}(BB)}_{=0} + \underbrace{V^{\text{ind}}(G\varnothing) - V^{\text{ind}}(B\varnothing)}_{=\frac{p_{\mathcal{S}}^{\text{ind}}(G\varnothing) - p_{\mathcal{S}}^{\text{ind}}(B\varnothing)}{2}}\Bigg). \end{aligned}$$

The difference $V_{\mathcal{S}}^{\text{ind}}(GB) - V_{\mathcal{S}}^{\text{ind}}(BB)$ vanishes because both the period payoff and continuation payoffs are the same for histories GB and BB as the firm does not invest following a bad outcome. $V_{\mathcal{S}}^{\text{ind}}(G\varnothing) - V_{\mathcal{S}}^{\text{ind}}(B\varnothing) = \frac{p_{\mathcal{S}}^{\text{ind}}(G\varnothing) - p_{\mathcal{S}}^{\text{ind}}(B\varnothing)}{2}$ because, for histories $G\varnothing$ and $B\varnothing$, the period profit is different, whereas the continuation payoff is the same. Also, note that $p_{\mathcal{S}}^{\text{ind}}(h_{-2}B) = \pi_L$ as even a competent firm does not invest after an outcome B. So, the equations above can be re-written as follows:

$$\begin{aligned} X &= \tfrac{p_{\mathcal{S}}^{\text{ind}}(GG) - \pi_L - c}{2} + \tfrac{\delta}{2} \cdot \left(\pi_H \cdot X - \pi_L \cdot Y + \tfrac{p_{\mathcal{S}}^{\text{ind}}(G\varnothing) - p_{\mathcal{S}}^{\text{ind}}(B\varnothing)}{2}\right) \\ Y &= \tfrac{p_{\mathcal{S}}^{\text{ind}}(BG) - \pi_L - c}{2} + \tfrac{\delta}{2} \cdot \left(\pi_H \cdot X - \pi_L \cdot Y + \tfrac{p_{\mathcal{S}}^{\text{ind}}(G\varnothing) - p_{\mathcal{S}}^{\text{ind}}(B\varnothing)}{2}\right). \end{aligned}$$

Solving for X and Y gives:

$$\begin{aligned} X &= \frac{\frac{-c + p_{\mathcal{S}}^{\text{ind}}(GG) - \pi_L}{2} + \frac{\delta\left(p_{\mathcal{S}}^{\text{ind}}(G\varnothing) - p_{\mathcal{S}}^{\text{ind}}(B\varnothing)\right)}{4} + \frac{\delta}{2}\pi_L \frac{p_{\mathcal{S}}^{\text{ind}}(GG) - p_{\mathcal{S}}^{\text{ind}}(BG)}{2}}{1 - \frac{\delta}{2}(\pi_H - \pi_L)}, \\ Y &= \frac{\frac{-c + p_{\mathcal{S}}^{\text{ind}}(BG) - \pi_L}{2} + \frac{\delta\left(p_{\mathcal{S}}^{\text{ind}}(G\varnothing) - p_{\mathcal{S}}^{\text{ind}}(B\varnothing)\right)}{4} + \frac{\delta}{2}\pi_H \frac{p_{\mathcal{S}}^{\text{ind}}(GG) - p_{\mathcal{S}}^{\text{ind}}(BG)}{2}}{1 - \frac{\delta}{2}(\pi_H - \pi_L)}. \end{aligned}$$

Analogous calculations with $Z := V_{\mathcal{S}}^{\text{ind}}(\varnothing G) - V_{\mathcal{S}}^{\text{ind}}(\varnothing B)$ yields $Z = X - \frac{p_{\mathcal{S}}^{\text{ind}}(GG) - p_{\mathcal{S}}^{\text{ind}}(\varnothing G)}{2} = Y - \frac{p_{\mathcal{S}}^{\text{ind}}(BG) - p_{\mathcal{S}}^{\text{ind}}(\varnothing G)}{2}$ Consequently, by Equation (8),

$$\begin{aligned}
\underline{C}_{\{G,\varnothing\}}^{\text{ind}} &= \frac{\delta(\pi_H - \pi_L)}{2} \cdot \Big(p_{\{G,\varnothing\}}^{\text{ind}}(BG) - \pi_L + \frac{\delta}{2}\big(p_{\{G,\varnothing\}}^{\text{ind}}(G\varnothing) - p_{\{G,\varnothing\}}^{\text{ind}}(B\varnothing)\big) + \frac{\delta}{2}\pi_H\big(p_{\{G,\varnothing\}}^{\text{ind}}(GG) - p_{\{G,\varnothing\}}^{\text{ind}}(BG)\big)\Big) \\
\underline{C}_{\{G\}}^{\text{ind}} &= \frac{\delta(\pi_H - \pi_L)}{2} \cdot \Big(p_{\{G\}}^{\text{ind}}(BG) - \pi_L + \frac{\delta}{2}\big(p_{\{G\}}^{\text{ind}}(G\varnothing) - p_{\{G\}}^{\text{ind}}(B\varnothing)\big) + \frac{\delta}{2}\pi_H\big(p_{\{G\}}^{\text{ind}}(GG) - p_{\{G\}}^{\text{ind}}(BG)\big)\Big) \\
&\quad - \max\left\{ \frac{\delta(\pi_H - \pi_L)}{2} \cdot \Big(1 - \frac{\delta}{2}(\pi_H - \pi_L)\Big) \Big(p_{\{G\}}^{\text{ind}}(BG) - p_{\{G\}}^{\text{ind}}(\varnothing G)\Big), 0 \right\} \\
\overline{C}_{\{G,\varnothing\}}^{\text{ind}} &= \frac{\delta(\pi_H - \pi_L)}{2} \cdot \Big(p_{\{G,\varnothing\}}^{\text{ind}}(GG) - \pi_L + \frac{\delta}{2}\big(p_{\{G,\varnothing\}}^{\text{ind}}(G\varnothing) - p_{\{G,\varnothing\}}^{\text{ind}}(B\varnothing)\big) + \frac{\delta}{2}\pi_L\big(p_{\{G,\varnothing\}}^{\text{ind}}(GG) - p_{\{G,\varnothing\}}^{\text{ind}}(BG)\big)\Big) \\
&\quad - \max\left\{ \frac{\delta(\pi_H - \pi_L)}{2} \cdot \Big(1 - \frac{\delta}{2}(\pi_H - \pi_L)\Big) \Big(p_{\{G,\varnothing\}}^{\text{ind}}(GG) - p_{\{G,\varnothing\}}^{\text{ind}}(\varnothing G)\Big), 0 \right\} \\
\overline{C}_{\{G\}}^{\text{ind}} &= \frac{\delta(\pi_H - \pi_L)}{2} \cdot \Big(p_{\{G\}}^{\text{ind}}(GG) - \pi_L + \frac{\delta}{2}\big(p_{\{G\}}^{\text{ind}}(G\varnothing) - p_{\{G\}}^{\text{ind}}(B\varnothing)\big) + \frac{\delta}{2}\pi_L\big(p_{\{G\}}^{\text{ind}}(GG) - p_{\{G\}}^{\text{ind}}(BG)\big)\Big)
\end{aligned}$$

In order to compute $\overline{C}_{\mathcal{S}}^{\text{ind}}$ and $\underline{C}_{\mathcal{S}}^{\text{ind}}$, we need to calculate the equilibrium prices. More precisely, for a given equilibrium strategy, we identify consumers' posterior beliefs and prices, and plug them into $\overline{C}_{\mathcal{S}}^{\text{ind}}$ and $\underline{C}_{\mathcal{S}}^{\text{ind}}$.

I–1.a) Existence of $\{G, \varnothing\}-$ *pure strategy equilibrium.*

If a firm plays a $\{G, \varnothing\}$-equilibrium with no investment after outcome B, then for a competent type the transition matrix of the Markov chain describing the evolution of outcomes is:

$$\begin{pmatrix} \frac{\pi_H}{2} & \frac{1}{2} & \frac{1-\pi_H}{2} \\ \frac{\pi_H}{2} & \frac{1}{2} & \frac{1-\pi_H}{2} \\ \frac{\pi_L}{2} & \frac{1}{2} & \frac{1-\pi_L}{2} \end{pmatrix}$$

Thus, the stationary distribution of outcomes for a competent firm is given by

$$\pi_{C,\{G,\varnothing\}}(G) = \frac{\pi_H + \pi_L}{2(2 - (\pi_H - \pi_L))},\ \pi_{C,\{G,\varnothing\}}(B) = \frac{1 - \pi_H}{2 - (\pi_H - \pi_L)},\ \pi_{C,\{G,\varnothing\}}(\varnothing) = \frac{1}{2}$$

For an incompetent type, the stationary distribution is $\pi_{I,\{G,\varnothing\}}(G) = \frac{\pi_L}{2}$, $\pi_{I,\{G,\varnothing\}}(B) = \frac{1-\pi_L}{2}$, and $\pi_{I,\{G,\varnothing\}}(\varnothing) = \frac{1}{2}$ because it never makes an investment.

Here, we focus on the prices if $\pi_L = 0$, i.e., when a G-outcome reveals that the firm is competent. Then, the posterior beliefs in this equilibrium, denoted by $\Pr^{\text{ind}}(h)$ are obtained by Bayes' rule. As only a competent firm invests after histories G and $\varnothing$, we have

$$\begin{aligned}
\Pr_{\{G,\varnothing\}}^{\text{ind}}(GG) =&\ \Pr_{\{G,\varnothing\}}^{\text{ind}}(G\varnothing) = \Pr_{\{G,\varnothing\}}^{\text{ind}}(GB) = 1, \\
\Pr_{\{G,\varnothing\}}^{\text{ind}}(\varnothing G) =&\ \tfrac{\mu\pi_H}{\mu\pi_H + (1-\mu)\pi_L} \to_{\pi_L \to 0} 1 \qquad\qquad \text{and} \\
\Pr_{\{G,\varnothing\}}^{\text{ind}}(BG) =&\ \Pr_{\{G,\varnothing\}}^{\text{ind}}(B\varnothing) = \Pr_{\{G,\varnothing\}}^{\text{ind}}(BB) = \tfrac{2\mu(1-\pi_H)}{2\mu(1-\pi_H) + (1-\mu)(2-\pi_H)}.
\end{aligned}$$

Given the posterior beliefs, $p^{\text{ind}}_{\{G,\varnothing\}}(\mathbf{h}) = \pi_H \cdot \Pr^{\text{ind}}_{\{G,\varnothing\}}(\mathbf{h})$ for $\mathbf{h} = (h_{-2}, s)$ for $s \in \{G, \varnothing\}$ and $p^{\text{ind}}_{\{G,\varnothing\}}(\mathbf{h}) = \pi_L$ otherwise. Thus, for $\pi_L = 0$:

$$\begin{aligned}
\overline{C}^{\text{ind}}_{\{G,\varnothing\}} &= \frac{\delta \cdot \pi_H^2}{2} \cdot \left(1 + \frac{\delta}{2}\left(1 - \frac{2\mu(1-\pi_H)}{2\mu(1-\pi_H) + (1-\mu)(2-\pi_H)}\right)\right) \\
\underline{C}^{\text{ind}}_{\{G,\varnothing\}} &= \frac{\delta \cdot \pi_H^2}{2} \cdot \left(\frac{2\mu(1-\pi_H)}{2\mu(1-\pi_H) + (1-\mu)(2-\pi_H)} + \frac{\delta}{2}(1+\pi_H)\left(1 - \frac{2\mu(1-\pi_H)}{2\mu(1-\pi_H) + (1-\mu)(2-\pi_H)}\right)\right).
\end{aligned}$$

Thus, we have

$$\overline{C}^{\text{ind}}_{\{G,\varnothing\}} - \underline{C}^{\text{ind}}_{\{G,\varnothing\}} = \frac{\delta \cdot \pi_H^2}{2} \cdot \left(1 - \frac{2\mu(1-\pi_H)}{2\mu(1-\pi_H) + (1-\mu)(2-\pi_H)}\right)\left(1 - \frac{\delta}{2}\pi_H\right) > 0,$$

but $\lim_{\mu\to 1} \overline{C}^{\text{ind}}_{\{G,\varnothing\}} = \lim_{\mu\to 1} \underline{C}^{\text{ind}}_{\{G,\varnothing\}} = \frac{\delta\pi_H^2}{2} > 0 = \lim_{\mu\to 1} \bar{c}^{\text{col}}$. Thus, for sufficiently large μ, the $\{G,\varnothing\}$-equilibrium only exists if c is sufficiently close to $\frac{\delta\pi_H^2}{2}$.

I–1.b) Existence of $\{G,\varnothing\}-$ equilibrium with mixing after B.

If the firm chooses to invest following an outcome B with probability $\alpha_B > 0$, then, the equilibrium beliefs change accordingly and the price after observing a last outcome B is no longer π_L. In particular, $p^{\text{ind}}_{\{G,\varnothing\}}(GB) > p^{\text{ind}}_{\{G,\varnothing\}}(BB) > \pi_L$ and $p^{\text{ind}}_{\{G,\varnothing\}}(GG) = p^{\text{ind}}_{\{G,\varnothing\}}(BG) = \pi_H$. It follows immediately that $\overline{C}^{\text{ind}}_{\{G,\varnothing\}} < \lim_{\mu\to 1} \underline{C}^{\text{ind}}_{\{G,\varnothing\}}$, i.e., there is no cost region such that this mixed equilibrium can exist.

I–1.c) Non-existence of $\{G\}-$ pure-strategy equilibrium.

If a firm plays a $\{G\}$-equilibrium, then for a competent type the transition matrix of the Markov chain describing the evolution of outcomes $\{G, \varnothing, B\}$ is given by

$$\begin{pmatrix} \frac{\pi_H}{2} & \frac{1}{2} & \frac{1-\pi_H}{2} \\ \frac{\pi_L}{2} & \frac{1}{2} & \frac{1-\pi_L}{2} \\ \frac{\pi_L}{2} & \frac{1}{2} & \frac{1-\pi_L}{2} \end{pmatrix}$$

Thus, the stationary distribution of outcomes for a competent firm is given by

$$\pi_{C,\{G,\varnothing\}}(G) = \frac{\pi_L}{2-(\pi_H-\pi_L)},\ \pi_{C,\{G,\varnothing\}}(B) = \frac{2-(\pi_H+\pi_L)}{2(2-(\pi_H-\pi_L))},\ \pi_{C,\{G,\varnothing\}}(\varnothing) = \frac{1}{2}$$

Because only a competent firm invests after histories G and no firm invests otherwise, we have

$$\Pr^{\text{ind}}_{\{G\}}(GG) = 1, \quad \Pr^{\text{ind}}_{\{G\}}(\varnothing G) = \mu \quad \text{and}$$
$$\Pr^{\text{ind}}_{\{G\}}(BG) = \frac{\mu(2-\pi_H-\pi_L)}{\mu(2-\pi_H-\pi_L)+(1-\mu)(2-\pi_H(1-\pi_L)-\pi_L(1+\pi_L))}.$$

Given the posterior beliefs, $p^{\text{ind}}_{\{G\}}(\mathbf{h}) = \pi_H \cdot \Pr^{\text{ind}}_{\{G\}}(\mathbf{h})$ for $\mathbf{h} = (h_{-2}, s)$ for $s \in \{G\}$ and

$p^{\text{ind}}_{\{G\}}(\mathbf{h}) = \pi_L$ otherwise. Thus, for $\pi_L = 0$:

$$\begin{aligned}
\overline{C}^{\text{ind}}_{\{G\}} &= \frac{\delta\cdot\pi_H^2}{2}\\
\underline{C}^{\text{ind}}_{\{G\}} &= \frac{\delta\cdot\pi_H^2}{2}\cdot\left(\frac{\mu(2-\pi_H)}{\mu(2-\pi_H)+(1-\mu)(2-\pi_H)}+\frac{\delta\pi_H}{2}\left(1-\frac{\mu(2-\pi_H)}{\mu(2-\pi_H)+(1-\mu)(2-\pi_H)}\right)\right.\\
&\quad\left.-(1-\frac{\delta\pi_H}{2})\left(\frac{\mu(2-\pi_H)}{\mu(2-\pi_H)+(1-\mu)(2-\pi_H)}-\mu\right)\right)\\
&= \frac{\delta\cdot\pi_H^2}{2}\cdot\left(\frac{\delta\pi_H}{2}+(1-\frac{\delta\pi_H}{2})\mu\right)
\end{aligned}$$

Thus, we have $\overline{C}^{\text{ind}}_{\{G\}} - \underline{C}^{\text{ind}}_{\{G\}} > 0$ but $\lim_{\mu\to1}\overline{C}^{\text{ind}}_{\{G\}} = \lim_{\mu\to1}\underline{C}^{\text{ind}}_{\{G\}} = \frac{\delta\pi_H^2}{2} > 0 = \lim_{\mu\to1}\overline{c}^{\text{col}}$. Thus, for sufficiently large μ, the $\{G\}$-equilibrium only exists if c is sufficiently close to $\frac{\delta\pi_H^2}{2}$.

I–1.d) Non-existence of $\{G\}-$ equilibrium with mixing after B and/or $\varnothing$.

As for the $\{G,\varnothing\}$- mixed equilibrium, if the firm invests after outcome B with positive probability, then $p^{\text{ind}}_{\{G,\varnothing\}}(GB) > p^{\text{ind}}_{\{G,\varnothing\}}(BB) > \pi_L$ and $p^{\text{ind}}_{\{G,\varnothing\}}(GG) = p^{\text{ind}}_{\{G,\varnothing\}}(BG) = \pi_H$. It follows immediately that $\overline{C}^{\text{ind}}_{\{G,\varnothing\}} < \underline{C}^{\text{ind}}_{\{G,\varnothing\}}$, i.e., there is no cost region such that such a mixed equilibrium can exist.

If the firm does not invest after B but wiht a positive probability after $\varnothing$ with a positive probability, then $p^{\text{ind}}_{\{G,\varnothing\}}(GG) = p^{\text{ind}}_{\{G,\varnothing\}}(\varnothing G) = \pi_H > p^{\text{ind}}_{\{G,\varnothing\}}(BG)$ and $p^{\text{ind}}_{\{G,\varnothing\}}(GB) = p^{\text{ind}}_{\{G,\varnothing\}}(BB) = p^{\text{ind}}_{\{G,\varnothing\}}(\varnothing B) = \pi_L$. Thus, an equilibrium in which a competent firm is indifferent between investing and not invest after $\varnothing$ and wants to invest after G exists as $\mu\to1$ if and only if c is close to $\frac{\delta\pi_H^2}{2}$.

I–2. Non-existence of $\{B,\varnothing\}-$ and $\{B\}-$equilibrium.

In these equilibria, a competent firm invests following a bad outcome, but not after a good outcome. This means $\delta(\pi_H-\pi_L)X < c < \delta(\pi_H-\pi_L)Y$, where as defined earlier, $X := V^{\text{ind}}_{\mathcal{S}}(GG) - V^{\text{ind}}_{\mathcal{S}}(GB)$, and $Y := V^{\text{ind}}_{\mathcal{S}}(BG) - V^{\text{ind}}_{\mathcal{S}}(BB)$.

Solving for X and Y and using $p^{\text{ind}}_{\mathcal{S}}(h_{-2}G) = \pi_L$ in this equilibrium yields

$$\begin{aligned}
X &= \frac{\frac{\pi_L-p^{\text{ind}}_{\mathcal{S}}(GB)+c}{2}+\frac{\delta}{2}\pi_H\frac{p^{\text{ind}}_{\mathcal{S}}(BB)-p^{\text{ind}}_{\mathcal{S}}(GB)}{2}+\frac{\delta}{2}\frac{p^{\text{ind}}_{\mathcal{S}}(GB)-p^{\text{ind}}_{\mathcal{S}}(BB)+p^{\text{ind}}_{\mathcal{S}}(G\varnothing)-p^{\text{ind}}_{\mathcal{S}}(B\varnothing)}{2}}{1+\frac{\delta}{2}(\pi_H-\pi_L)}\\
Y &= \frac{\frac{\pi_L-p^{\text{ind}}_{\mathcal{S}}(BB)+c}{2}+\frac{\delta}{2}\pi_L\frac{p^{\text{ind}}_{\mathcal{S}}(BB)-p^{\text{ind}}_{\mathcal{S}}(GB)}{2}+\frac{\delta}{2}\frac{p^{\text{ind}}_{\mathcal{S}}(GB)-p^{\text{ind}}_{\mathcal{S}}(BB)+p^{\text{ind}}_{\mathcal{S}}(G\varnothing)-p^{\text{ind}}_{\mathcal{S}}(B\varnothing)}{2}}{1+\frac{\delta}{2}(\pi_H-\pi_L)}
\end{aligned}$$

So, the condition $c < \delta(\pi_H-\pi_L)Y$ holds if and only if

$$c < \overline{C}_{\mathcal{S}} := \pi_L - p^{\text{ind}}_{\mathcal{S}}(BB) + \frac{\delta}{2}(1-\pi_L)(p^{\text{ind}}_{\mathcal{S}}(GB) - p^{\text{ind}}_{\mathcal{S}}(BB)) + \frac{\delta}{2}(p^{\text{ind}}_{\mathcal{S}}(G\varnothing) - p^{\text{ind}}_{\mathcal{S}}(B\varnothing))$$

for $\mathcal{S} = \{B\}$, or $\{B,\varnothing\}$.

I–2.a) Non-existence of $\{B,\varnothing\}-$ pure-strategy equilibrium.

If the competent type plays a $\{B,\varnothing\}-$equilibrium, then the evolution of outcomes produced by the competent firm, $\{G,\varnothing,B\}$, is captured by the following transition matrix:

$$\begin{pmatrix} \frac{\pi_L}{2} & \frac{1}{2} & \frac{1-\pi_L}{2} \\ \frac{\pi_H}{2} & \frac{1}{2} & \frac{1-\pi_H}{2} \\ \frac{\pi_H}{2} & \frac{1}{2} & \frac{1-\pi_H}{2} \end{pmatrix}$$

Therefore, the stationary distribution of outcomes is as follows:

$$\pi_{C,\{B,\varnothing\}}(G) = \frac{\pi_H}{2+\pi_H-\pi_L},\ \pi_{C,\{B,\varnothing\}}(B) = \frac{2-(\pi_H+\pi_L)}{2(2+\pi_H-\pi_L)},\ \pi_{C,\{B,\varnothing\}}(\varnothing) = \frac{1}{2}$$

For an incompetent type, the stationary distribution is $\pi_{I,\mathcal{S}}(G) = \frac{\pi_L}{2}$, $\pi_{I,\mathcal{S}}(B) = \frac{1-\pi_L}{2}$, and $\pi_{I,\mathcal{S}}(\varnothing) = \frac{1}{2}$ because it never makes an investment. Given this stationary distribution of outcomes by the competent type, and that by the incompetent type, we compute the posterior beliefs using Bayes rule.

For $\pi_L = 0$, we see that a good outcome can only be produced by the competent type. At the same time, following a good outcome G, both types make the same decision and do not invest. This implies

$$\begin{aligned} &\Pr^{\text{ind}}_{\{B,\varnothing\}}(GB) = \Pr^{\text{ind}}_{\{B,\varnothing\}}(G\varnothing) = 1, \quad && \Pr^{\text{ind}}_{\{B,\varnothing\}}(BB) = \frac{\mu(2-\pi_H)(1-\pi_H)}{2+\pi_H-4\mu\pi_H+\mu\pi_H^2} \\ &\Pr^{\text{ind}}_{\{B,\varnothing\}}(B\varnothing) = \frac{\mu(2-\pi_H)}{2+(1-2\mu)\pi_H}, && \Pr^{\text{ind}}_{\{B,\varnothing\}}(\varnothing B) = \frac{\mu(1-\pi_H)}{1-\mu\pi_H} \end{aligned}$$

Given the posterior beliefs, $p^{\text{ind}}_{\{B,\varnothing\}}(\mathbf{h}) = \pi_H \cdot \Pr^{\text{ind}}_{\{B,\varnothing\}}(\mathbf{h})$ for $\mathbf{h} = (h_{-2}, s)$ for $s \in \{B,\varnothing\}$ and $p^{\text{ind}}_{\{B,\varnothing\}}(\mathbf{h}) = \pi_L$ otherwise. Then, $\lim_{\mu\to 1} \overline{C}_{\{B,\varnothing\}} = -\pi_H < 0$. Therefore, for any $c > 0$, the condition $c < \overline{C}_{\{B,\varnothing\}}$ cannot be true.

I–2.b) Non-existence of $\{B,\varnothing\}-$ mixed-strategy equilibrium.

If a competnet firm invests after an outcome G with positive probability α_G, then $p_{\{B,\varnothing\}}(GG) = p_{\{B,\varnothing\}}(BG) = \alpha_G\pi_H$ and $p_{\{B,\varnothing\}}(GB) = \pi_H > p_{\{B,\varnothing\}}(BB)$. Thus, it is straight forward to show that as $\mu \to 1$ after a G outcome the firm has no incentive to invest. Hence, such a mixed strategy equilibrium can also not exist.

I–2.c) Non-existence of $\{B\}-$ pure-strategy equilibrium.

For the $\{B\}-$equilibrium, the evolution of outcomes produced by an outcome is captured by the transition matrix:

$$\begin{pmatrix} \frac{\pi_L}{2} & \frac{1}{2} & \frac{1-\pi_L}{2} \\ \frac{\pi_L}{2} & \frac{1}{2} & \frac{1-\pi_L}{2} \\ \frac{\pi_H}{2} & \frac{1}{2} & \frac{1-\pi_H}{2} \end{pmatrix}$$

Therefore, the stationary distribution of outcomes is as follows:

$$\pi_{C,\{B\}}(G) = \frac{\pi_H + \pi_L}{2(2+\pi_H-\pi_L)},\ \pi_{C,\{B\}}(B) = \frac{1-\pi_L}{2+\pi_H-\pi_L},\ \pi_{C,\{B\}}(\varnothing) = \frac{1}{2}$$

Based on these stationary distribution of outcomes, we can compute posterior beliefs for $\pi_L = 0$: $\Pr^{\text{ind}}_{\{B\}}(GB) = \frac{\pi_H}{2(2+\pi_H)}$ and $\Pr^{\text{ind}}_{\{B\}}(BB) = \frac{2\mu(1-\pi_H)}{2+\pi_H(1-3\mu)}$. Given the posterior beliefs, $p^{\text{ind}}_{\{B\}}(\mathbf{h}) = \pi_H \cdot \Pr^{\text{ind}}_{\{B\}}(\mathbf{h})$ for $\mathbf{h} = (h_{-2}, s)$ for $s \in \{B\}$ and $p^{\text{ind}}_{\{B\}}(\mathbf{h}) = \pi_L$ otherwise. Thus, $\overline{C}_{\{B\}} = -p^{\text{ind}}_{\mathcal{S}}(BB) + \frac{\delta}{2}(1-\pi_L)(p^{\text{ind}}_{\mathcal{S}}(GB) - p^{\text{ind}}_{\mathcal{S}}(BB))$, which converges to $-\pi_H - \frac{\pi_H\delta(4+\pi_H)}{4(2+\pi_H)} < 0$ as μ approaches 1. Therefore, the condition $c < \overline{C}_{\{B\}}$ cannot hold.

I–2.d) Non-existence of $\{B\}-$ mixed-strategy equilibrium.

A combination of the arguments above easily shows that the incentive to invest after G as $\mu \to 1$ is negative. Thus, there cannot be a $\{B\}-$ mixed-strategy equilibrium with a positive weight on G. Similarly, if the firm invested with positive probability after $\varnothing$, then an analogous argument to the non-existence of a pure-strategy $\{B, \varnothing\}$-equilibrium shows that this equilibrium cannot exist.

I–3. Non-existence of $\{G, B\}-$ and $\{\varnothing\}-$equilibrium.

It remains to examine two more equilibria: $\mathcal{S} = \{G, B\}$ and $\{\varnothing\}$. These equilibria demonstrate strategies non-monotonic in the firm's reputation in the sense that the firm takes the same action following a good and bad outcome, but a different one following an empty outcome. The argument is the same for mixed-equilibria, so we omit the argument here.

First, for $\mathcal{S} = \{G, B\}$, the firm must find it optimal to invest following a good and bad outcome, but not after an $\varnothing$-outcome. Then, the equilibrium exists if and only if

$$\delta(\pi_H - \pi_L)\cdot(V^{\text{ind}}_{\{G,B\}}(\varnothing G) - V^{\text{ind}}_{\{G,B\}}(\varnothing B)) \le c \le \min_{x\in\{G,B\}} \big(\delta(\pi_H - \pi_L)\cdot(V^{\text{ind}}_{\{G,B\}}(xG) - V^{\text{ind}}_{\{G,B\}}(xB))\big).$$

The future payoffs of $V^{\text{ind}}_{\{G,B\}}(yG)$ and $V^{\text{ind}}_{\{G,B\}}(yB)$ are exactly the same for any $y \in \{G, B, \varnothing\}$. So, the difference in these payoff functions is equal to the difference in immediate per-period profit. Hence, a $\{G, B\}$ equilibrium can exist for c such that

$$\frac{\delta(\pi_H - \pi_L)}{2}\cdot\big(p^{\text{ind}}_{\{G,B\}}(\varnothing G) - p^{\text{ind}}_{\{G,B\}}(\varnothing B)\big) \le c \le \min_{x\in\{G,B\}} \frac{\delta(\pi_H - \pi_L)}{2}\cdot\big(p^{\text{ind}}_{\{G,B\}}(xG) - p^{\text{ind}}_{\{G,B\}}(xB)\big).$$

If $\pi_L = 0$, the right-hand side is zero for $x = G$, as one good outcome reveals the firm to be competent. Thus, the equilibrium does not exist for any strictly positive c.

We can similarly show that the $\{\varnothing\}-$equilibrium does not exist. If it did, the competent type would invest following an $\varnothing$ outcome, but not after a good or bad one. So, the following condition must hold:

$$\max_{x\in\{G,B\}} \frac{\delta(\pi_H-\pi_L)}{2}\cdot\big(\underbrace{p^{\text{ind}}_{\{\varnothing\}}(xG)-p^{\text{ind}}_{\{\varnothing\}}(xB)}_{=\pi_L-\pi_L=0}\big)\le c\le \frac{\delta(\pi_H-\pi_L)}{2}\cdot\big(\underbrace{p^{\text{ind}}_{\{\varnothing\}}(\varnothing G)-p^{\text{ind}}_{\{\varnothing\}}(\varnothing B)}_{=\pi_L-\pi_L=0}\big). \tag{9}$$

Both the left-hand and the right-hand side vanish, and the equilibrium does not exist for any strictly positive c.

Summing up the previous analysis from **I–1** to **I–3** above, we conclude that if $\pi_L = 0$ and μ is sufficiently large, and if $c > \overline{c}^{\text{ind}}$, the no investment equilibrium is the unique equilibrium for an individual brand except for in a small neighborhood around $c = \frac{\delta\pi_H^2}{2}$ where the $\{G\}$ and $\{G, \varnothing\}$ exist.

This implies that whenever the reputational equilibrium does not exist for an individual brand, the competent type never makes investment, and consumers pay the minimal price. This suggests that if $c > \overline{c}^{\text{ind}}$, collective branding would be an attractive option for the firm as long as it provides more commitment power, i.e., it admits a more profitable equilibrium, such as the reputational equilibrium.

Next, we move on to collective branding and prove the statement regarding a collective brand in Proposition 5.

II. Collective brand:

If $c > \overline{c}^{\text{col}}$, the reputational equilibrium does not exist for a collective brand. Besides the reputational and no investment equilibrium, there are two alternative equilibria: $\mathcal{S} = \{G\}$, or $\{B\}$. The payoff of a firm in a collective brand depends on the type of the other firm, $\theta \in \{C, I\}$. Whenever we focus on one specific equilibrium, we omit the notation $\mathcal{S}$. Similar to the analysis for individual brands, such an equilibrium can only be sustained if the cost of investment is neither too large nor too small. Formally, an $\mathcal{S}-$equilibrium exists if and only if $\underline{C}^{\text{col}}_{\mathcal{S}}(\theta) < c < \overline{C}^{\text{col}}_{\mathcal{S}}(\theta)$, where $\underline{C}^{\text{col}}_{\mathcal{S}}(\theta) := \delta\cdot(\pi_H-\pi_L)(V^{\text{col}}_{\mathcal{S}}(s'G;\theta) - V^{\text{col}}_{\mathcal{S}}(s'B;\theta))$ for $s' \notin \mathcal{S}$ and $\overline{C}^{\text{col}}_{\mathcal{S}}(\theta) := \delta\cdot(\pi_H-\pi_L)(V^{\text{col}}_{\mathcal{S}}(sG;\theta) - V^{\text{col}}_{\mathcal{S}}(sB;\theta))$ for $s \in \mathcal{S}$.

For all $s \in \mathcal{S}$, we can write

$$\begin{aligned} V^{\text{col}}_{\mathcal{S}}(h_{-2}s;C) &= \frac{p^{\text{col}}_{\mathcal{S}}(h_{-2}s)-c}{2} + \delta(\pi_H V^{\text{col}}_{\mathcal{S}}(sG;\theta) + (1-\pi_H)V^{\text{col}}_{\mathcal{S}}(sB;\theta)) \\ V^{\text{col}}_{\mathcal{S}}(h_{-2}s;I) &= \frac{p^{\text{col}}_{\mathcal{S}}(h_{-2}s)-c}{2} + \delta\left(\frac{\pi_H+\pi_L}{2}V^{\text{col}}_{\mathcal{S}}(sG;\theta) + \frac{(1-\pi_H)+(1-\pi_L)}{2}V^{\text{col}}_{\mathcal{S}}(sB;\theta)\right) \end{aligned}$$

and for all $s \notin \mathcal{S}$ and $\theta \in \{C, I\}$

$$V^{\text{col}}_{\mathcal{S}}(h_{-2}s;\theta) = \frac{p^{\text{col}}_{\mathcal{S}}(h_{-2}s)}{2} + \delta(\pi_L V^{\text{col}}_{\mathcal{S}}(sG;\theta) + (1-\pi_L)V^{\text{col}}_{\mathcal{S}}(sB;\theta))$$

We identify conditions under which a competent type finds it optimal to follow a given equilibrium strategy.

II–1. $\{G\}$–equilibrium

In an equilibrium with $\mathcal{S} = \{G\}$, following a good history, a firm finds it optimal to invest in quality, but not following a bad history. Thus, $p^{\text{col}}_{\{G\}}(h_{-2}B) = \pi_L$ and $V^{\text{col}}_{\{G\}}(GB;\theta) = V^{\text{col}}_{\{G\}}(BB;\theta)$.

Then, for $\theta = C$

$$\begin{aligned}V^{\text{col}}_{\{G\}}(h_{-2}G;C) - V^{\text{col}}_{\{G\}}(h_{-2}B;C) &= \frac{p^{\text{col}}_{\{G\}}(h_{-2}G) - \pi_L - c}{2} + \delta \cdot \big(\pi_H (V^{\text{col}}_{\{G\}}(GG;C) - V^{\text{col}}_{\{G\}}(GB;C)) \\ &\quad -\pi_L \cdot (V^{\text{col}}_{\{G\}}(BG;C) - V^{\text{col}}_{\{G\}}(BB;C))\big).\end{aligned}$$

Thus, we can calculate $V^{\text{col}}_{\{G\}}(GG;C) - V^{\text{col}}_{\{G\}}(GB;C)$ and $V^{\text{col}}_{\{G\}}(GB;C) - V^{\text{col}}_{\{G\}}(BG;C)$ to be

$$\begin{aligned}V^{\text{col}}_{\{G\}}(GG;C) - V^{\text{col}}_{\{G\}}(GB;C) &= \frac{p^{\text{col}}_{\{G\}}(GG) - \pi_L - c + \delta\pi_L(p^{\text{col}}_{\{G\}}(GG) - p^{\text{col}}_{\{G\}}(BG))}{2(1-\delta(\pi_H - \pi_L))} \\ V^{\text{col}}_{\{G\}}(BG;C) - V^{\text{col}}_{\{G\}}(BB;C) &= \frac{p^{\text{col}}_{\{G\}}(BG) - \pi_L - c + \delta\pi_H(p^{\text{col}}_{\{G\}}(GG) - p^{\text{col}}_{\{G\}}(BG))}{2(1-\delta(\pi_H - \pi_L))}.\end{aligned}$$

Thus,

$$\begin{aligned}\overline{C}^{\text{col}}_{\{G\}}(C) &= \delta \cdot (\pi_H - \pi_L) \cdot \frac{p^{\text{col}}_{\{G\}}(GG) - \pi_L + \delta\pi_L(p^{\text{col}}_{\{G\}}(GG) - p^{\text{col}}_{\{G\}}(BG))}{2(1-\frac{\delta}{2}(\pi_H - \pi_L))} \\ \underline{C}^{\text{col}}_{\{G\}}(C) &= \delta \cdot (\pi_H - \pi_L) \cdot \frac{p^{\text{col}}_{\{G\}}(BG) - \pi_L + \delta\pi_H(p^{\text{col}}_{\{G\}}(GG) - p^{\text{col}}_{\{G\}}(BG))}{2(1-\frac{\delta}{2}(\pi_H - \pi_L))}.\end{aligned}$$

If the other firm is incompetent, then

$$\begin{aligned}V^{\text{col}}_{\{G\}}(h_{-2}G;I) - V^{\text{col}}_{\{G\}}(h_{-2}B;I) &= \frac{p^{\text{col}}_{\{G\}}(h_{-2}G) - \pi_L - c}{2} + \delta \cdot \big(\frac{\pi_H + \pi_L}{2}(V^{\text{col}}_{\{G\}}(GG;I) - V^{\text{col}}_{\{G\}}(GB;I) \\ &\quad -\pi_L(V^{\text{col}}_{\{G\}}(BG;I) - V^{\text{col}}_{\{G\}}(BB;I)\big)\end{aligned}$$

yielding

$$\begin{aligned}V^{\text{col}}_{\{G\}}(GG;I) - V^{\text{col}}_{\{G\}}(GB;I) &= \frac{p^{\text{col}}_{\{G\}}(GG) - \pi_L - c + \delta\pi_L(p^{\text{col}}_{\{G\}}(GG) - p^{\text{col}}_{\{G\}}(BG))}{2(1-\delta \cdot \frac{\pi_H - \pi_L}{2})} \\ V^{\text{col}}_{\{G\}}(BG;I) - V^{\text{col}}_{\{G\}}(BB;I) &= \frac{p^{\text{col}}_{\{G\}}(BG) - \pi_L - c + \delta\frac{\pi_H + \pi_L}{2}(p^{\text{col}}_{\{G\}}(GG) - p^{\text{col}}_{\{G\}}(BG))}{2(1-\delta \cdot \frac{\pi_H - \pi_L}{2})}.\end{aligned}$$

Thus,

$$\begin{aligned}
\overline{C}^{\text{col}}_{\{G\}}(I) &= \frac{\delta}{2}\cdot(\pi_H-\pi_L)\cdot\big(p^{\text{col}}_{\{G\}}(GG)-\pi_L+\delta\pi_L(p^{\text{col}}_{\{G\}}(GG)-p^{\text{col}}_{\{G\}}(BG))\big)\\
\underline{C}^{\text{col}}_{\{G\}}(I) &= \frac{\delta}{2}\cdot(\pi_H-\pi_L)\cdot\big(p^{\text{col}}_{\{G\}}(BG)-\pi_L+\delta\frac{\pi_H+\pi_L}{2}(p^{\text{col}}_{\{G\}}(GG)-p^{\text{col}}_{\{G\}}(BG))\big).
\end{aligned}$$

Next, we calculate the equilibrium prices. If both firms are competent, the transition matrix for a competent firm between outcomes $\{G,B\}$ is given by

$$\begin{pmatrix} \pi_H & 1-\pi_H \\ \pi_L & 1-\pi_L \end{pmatrix}.$$

Thus, the stationary probability of being in state G is $\pi_{CC}(G)=\frac{\pi_L}{\pi_L+1-\pi_H}$ and the probability of being in state B is $\pi_{CC}(B)=\frac{1-\pi_H}{\pi_L+1-\pi_H}$. If one is competent and the other is incompetent, then it is given by

$$\begin{pmatrix} \frac{\pi_H+\pi_L}{2} & 1-\frac{\pi_H+\pi_L}{2} \\ \pi_L & 1-\pi_L \end{pmatrix}.$$

Thus, the stationary probability of being in state G is $\pi_{CI}(G)=\frac{\pi_L}{\frac{\pi_L}{2}+1-\frac{\pi_H}{2}}$ and the probability of being in state B is $\pi_{CI}(B)=\frac{1-\frac{\pi_H+\pi_L}{2}}{\frac{\pi_L}{2}+1-\frac{\pi_H}{2}}$. If both are incompetent, then $\pi_{II}(G)=\pi_L$ and the probability of being in state B is $\pi_{II}(B)=1-\pi_L$.

For $\pi_L=0$, after observing a history GG, a consumer updates his belief about facing a competent firm by Bayes' rule to $\Pr(GG)=\frac{1+\mu-\pi_H}{2-(2-\mu)\pi_H}$. And, after observing a history BG, a consumer updates his belief about facing a competent firm is $\Pr(BG)=\mu$. Thus, given $\pi_L=0$,

$$\begin{aligned}
\overline{C}^{\text{col}}_{\{G\}}(C) &= \delta\cdot\pi_H\cdot\frac{p^{\text{col}}_{\{G\}}(GG)}{2(1-\frac{\delta}{2}\pi_H)}\rightarrow_{\mu\rightarrow1}\frac{\delta\cdot\pi_H^2}{2(1-\frac{\delta}{2}\pi_H)}\\
\underline{C}^{\text{col}}_{\{G\}}(C) &= \delta\cdot\pi_H\cdot\frac{p^{\text{col}}_{\{G\}}(BG)+\delta\pi_H(p^{\text{col}}_{\{G\}}(GG)-p^{\text{col}}_{\{G\}}(BG))}{2(1-\frac{\delta}{2}\pi_H)}\rightarrow_{\mu\rightarrow1}\frac{\delta\cdot\pi_H^2}{2(1-\frac{\delta}{2}\pi_H)}\\
\overline{C}^{\text{col}}_{\{G\}}(I) &= \frac{\delta}{2}\cdot\pi_H\cdot p^{\text{col}}_{\{G\}}(GG)\rightarrow_{\mu\rightarrow1}\delta\cdot\pi_H^2>\frac{\delta\cdot\pi_H^2}{2(1-\frac{\delta}{2}\pi_H)}\\
\underline{C}^{\text{col}}_{\{G\}}(I) &= \frac{\delta}{2}\cdot\pi_H\cdot\big(p^{\text{col}}_{\{G\}}(BG)+\delta\frac{\pi_H}{2}(p^{\text{col}}_{\{G\}}(GG)-p^{\text{col}}_{\{G\}}(BG))\big)\rightarrow_{\mu\rightarrow1}\delta\cdot\pi_H^2.
\end{aligned}$$

Thus, a $\{G\}$-equilibrium cannot exist for any costs c as $\mu\rightarrow1$ because a competent firm that is paired with either a competent or an incompetent firm always has an incentive to deviate from the equilibrium investment decision. Analogously, there cannot be a mixed equilibrium.

II–2. $\{B\}-$equilibrium

Similarly, one can show that no $\{B\}-$equilibrium can exist for sufficiently large μ.

Therefore, for $\pi_L = 0$ and μ close to 1, if $c > \overline{c}^{\text{col}}$, then no investment equilibrium is the unique equilibrium. This proves the proposition.

Step 2: Mean expected per-period profit

We have established that for $c \in (\overline{c}^{\text{ind}}, \overline{c}^{\text{col}})$, the only equilibrium for an individual brand is the "no investment" equilibrium. In this equilibrium, a competent firm's mean expected per-period profits are given by $\Pi^{\text{ind}} = \pi_L = 0$. In a collective brand, regardless of the other firm's competency, the firm's average profit in a reputation equlibrium is given by $\lim_{\mu \to 1} \Pi^{\text{col}} = \frac{1}{2}(\pi_H - c)$. Therefore, for sufficiently large μ the firm always prefers branding with another firm to staying alone as long as $c < \pi_H$ which is the case by assumption.

For $c \in (0, \min\{\overline{c}^{\text{ind}}, \overline{c}^{\text{col}}\})$, the reputational equilibrium exists for an individual and collective brand. Thus, after any history, consumers expect competent firms to invest, but the belief updating after a particular history is different. An individual firm makes an average profit of

$$\begin{aligned}\Pi^{\text{ind}} &= 0.25 \cdot \big(\pi_H^2 p^{\text{ind}}(GG) + 2 \cdot \pi_H(1-\pi_H)p^{\text{ind}}(GB) + (1-\pi_H)^2 p^{\text{ind}}(BB) + \\ &\quad 2 \cdot \pi_H p^{\text{ind}}(G\varnothing) + 2 \cdot (1-\pi_H)p^{\text{ind}}(B\varnothing) + p^{\text{ind}}(\varnothing\varnothing)\big) - c\end{aligned}$$

A competent firm forms a brand with another competent firm makes an average profit of

$$\Pi^{\text{col}} = \pi_H^2 p^{\text{col}}(GG) + \pi_H(1-\pi_H)p^{\text{col}}(GB) + \pi_H(1-\pi_H)p^{\text{col}}(BG) + (1-\pi_H)^2 p^{\text{col}}(BB) - c.$$

Then, as $\mu \to 1$ the difference in average profits satisfies

$$\lim_{\mu \to 1} \frac{\Pi^{\text{ind}} - \Pi^{\text{col}}}{(1-\mu)^2} = \frac{\pi_H^3(\pi_H(\pi_H((0.125\pi_H - 0.5)\pi_H + 0.75) - 0.5) + 0.125)}{(1-\pi_H)^6} > 0.$$

Thus, for large μ, a firm always prefers to stay alone to branding with another firm. Note that branding with an incompetent firm is always less attractive than branding with a competent firm.

When $c > \overline{c}^{\text{col}} > \overline{c}^{\text{ind}}$, the $\{G\}$ or $\{G, \varnothing\}$ exist for an individual brand for c close to $\frac{\delta\pi_H^2}{2}$ and in that case the profits are given by $\lim_{\mu \to 1} \Pi^{\text{ind}}_{\{G,\varnothing\}} = \frac{\pi_H^2 - c}{2(2-\pi_H)} > 0$. So, a competent type is better off joining a collective brand, even if the other firm is of incompetent type. For any c outside this small neighborhood around $\frac{\delta\pi_H^2}{2}$, only the "no investment" equilibrium exists, i.e., $\Pi^{\text{ind}}_{\{G,\varnothing\}} = 0$. Therefore, the firm is indifferent between individual and collective brand.

□

Proof. [Proposition 6] See Appendix B. □

Proof. [Proposition 7] Note that the history that must minimizes the benefit from investment is $h_{t-1} = GG$ and

$$\bar{c}_n^{\text{ind}}(GG) \equiv \pi_H^2 \frac{\delta^2}{n^2}\left[(1-\pi_H)\frac{1-\mu}{\mu(1-\pi_H)^2+1-\mu} + (n-1)\cdot\frac{1-\mu}{\mu(1-\pi_H)+1-\mu}\right].$$

Thus, we can write

$$\lim_{\mu\to 1}\frac{\bar{c}_n^{\text{ind}}(GG)}{1-\mu} = \frac{\pi_H^2\cdot\delta^2}{n}\frac{1}{(1-\pi_H)}$$

In the case of collective branding, given $\pi_L = 0$ the probability of facing a C-firm after a history $\mathbf{h}$ with u G-observations and $2-u$ B-observations simplifies to

$$\Pr^{\text{col}}(\mathbf{h}) = \frac{\sum_{i=1}^{n}\binom{n}{i}\mu^i(1-\mu)^{n-i}\sum_{v=u}^{2}\binom{2-u}{v-u}\left(\frac{i}{n}\right)^{v+1}\pi_H^u(1-\pi_H)^{v-u}\left(\frac{n-i}{n}\right)^{2-v}}{\sum_{i=0}^{n}\binom{n}{i}\mu^i(1-\mu)^{n-i}\sum_{v=u}^{2}\binom{2-u}{v-u}\left(\frac{i}{n}\right)^{v}\pi_H^u(1-\pi_H)^{v-u}\left(\frac{n-i}{n}\right)^{2-v}}$$

This equation is Bayes' rule. The denominator is the total probability that a history h is produced. Provided that i of n firms are competent, the second summation is the probability that u good and $2-u$ bad outcomes are produced. With $\pi_L = 0$, only competent type can produce a good outcome. $\left(\frac{i}{n}\right)^v\cdot\pi_H^u\cdot(1-\pi_H)^{v-u}$ is the probability that a competent type is drawn $v \geq u$ times and produce u good and $v-u$ bad outcomes. The remaining $2-v$ bad outcomes are generated if an incompetent type is drawn, which happens with probability $\left(\frac{n-i}{n}\right)^{2-v}$. Summing this over i gives the total probability. On the numerator is simply a joint probability that the collective brand produces h and a randomly drawn firm is competent. Therefore, there has to exist at least one competent type, which is represented in the lower bound $i = 1$ in the first sum. An additional factor of $\frac{i}{n}$ in the second summation completes the expression.

Now we can calculate the price differences:

$$\Pr^{\text{col}}(GG) - \Pr^{\text{col}}(GB) =$$

$$\sum_{i=1}^{n} \binom{n}{i} \mu^i (1-\mu)^{n-i} \sum_{j=1}^{n} \binom{n}{j} \mu^j (1-\mu)^{n-j}$$

$$\frac{\pi_H^2 \left(\frac{i}{n}\right)^3 \pi_H (\frac{j}{n}\left(\frac{n-j}{n}\right) + \left(\frac{j}{n}\right)^2 (1-\pi_H)) - \pi_H (\left(\frac{i}{n}\right)^2 \left(\frac{n-i}{n}\right) + \left(\frac{i}{n}\right)^3 (1-\pi_H)) \pi_H^2 \left(\frac{j}{n}\right)^2}{\left(\sum_{j=1}^{n} \binom{n}{j} \mu^j (1-\mu)^{-j} \pi_H^2 \left(\frac{j}{n}\right)^2\right) \left(\sum_{j=1}^{n} \binom{n}{j} \mu^j (1-\mu)^{n-j} \pi_H (\frac{j}{n}\left(\frac{n-j}{n}\right) + \left(\frac{j}{n}\right)^2 (1-\pi_H))\right)} =$$

$$\sum_{i=1}^{n} \binom{n}{i} \mu^i (1-\mu)^{n-i} \sum_{j=1}^{n} \binom{n}{j} \mu^j (1-\mu)^{n-j} \pi_H^3 \left(\frac{i}{n}\right)^2 \frac{j}{n}$$

$$\frac{(\frac{i}{n}\left(\frac{n-j}{n}\right) + \frac{i}{n}\frac{j}{n}(1-\pi_H)) - (\frac{j}{n}\left(\frac{n-i}{n}\right) + \frac{i}{n}\frac{j}{n}(1-\pi_H))}{\left(\sum_{j=1}^{n} \binom{n}{j} \mu^j (1-\mu)^{n-j} \pi_H^2 \left(\frac{j}{n}\right)^2\right) \left(\sum_{j=1}^{n} \binom{n}{j} \mu^j (1-\mu)^{n-j} \pi_H (\frac{j}{n}\left(\frac{n-j}{n}\right) + \left(\frac{j}{n}\right)^2 (1-\pi_H))\right)} =$$

$$\sum_{i=1}^{n} \sum_{j=1}^{n} \binom{n}{j} \binom{n}{i} \mu^{i+j} (1-\mu)^{2n-i-j} \pi_H^3 \left(\frac{i}{n}\right)^2 \frac{j}{n} \frac{\frac{i}{n} - \frac{j}{n}}{\pi(GG)\pi(GB)}$$

and consequently

$$\lim_{\mu \to 1} \frac{1}{1-\mu} (\Pr^{\text{col}}(GG) - \Pr^{\text{col}}(GB)) = \pi_H^3 \frac{n-1}{n^2} \frac{1}{\pi_H^3 (1-\pi_H)} = \frac{n-1}{n^2(1-\pi_H)}$$

Similarly, $(\Pr^{\text{col}}(GB) - \Pr^{\text{col}}(BB)) =$

$$\sum_{i=1}^{n} \binom{n}{i} \sum_{j=0}^{n} \binom{n}{j} \mu^{i+j} (1-\mu)^{2n-i-j} \pi_H \frac{i}{n}$$

$$\frac{\left(\left(\frac{i}{n}\right) \frac{n-i}{n} + \left(\frac{i}{n}\right)^2 (1-\pi_H)\right) \left(\left(\frac{n-j}{n}\right)^2 + 2\left(\frac{j}{n}\right) \frac{n-j}{n}(1-\pi_H) + \left(\frac{j}{n}\right)^2 (1-\pi_H)^2\right)}{\pi(GB)\pi(BB)}$$

$$-\frac{\left(\left(\frac{j}{n}\right) \frac{n-j}{n} + \left(\frac{j}{n}\right)^2 (1-\pi_H)\right) \left(\left(\frac{n-i}{n}\right)^2 + 2\left(\frac{i}{n}\right) \frac{n-i}{n}(1-\pi_H) + \left(\frac{i}{n}\right)^2 (1-\pi_H)^2\right)}{\pi(GB)\pi(BB)}$$

and consequently

$$\lim_{\mu \to 1} \frac{1}{1-\mu} (\Pr^{\text{col}}(GB) - \Pr^{\text{col}}(C|BB)) = \frac{n + \pi_H - n\pi_H}{n^2(1-\pi_H)^2}$$

Thus, the cutoff for $\theta = C$ and $h = GG$ can be written as

$$\begin{aligned}
\lim_{\mu \to 1} \frac{1}{1-\mu} \bar{c}^{\text{col}}(GG, C) &= \pi_H^2 \left[\left(\frac{\delta}{n} + \frac{\delta^2}{n} \pi_H \right) \frac{n-1}{n^2(1-\pi_H)} + \frac{\delta^2}{n}(1-\pi_H) \frac{n-(n-1)\pi_H}{n^2 \pi_H (1-\pi_H)^2} \right] \\
&= \frac{\delta(n-1+\delta n)\pi_H^2}{n^3(1-\pi_H)}
\end{aligned}$$

Thus, we can write

$$\begin{aligned}
\lim_{\mu \to 1} \frac{1}{1-\mu} (\bar{c}^{\text{col}}(GG, C) - \bar{c}^{\text{ind}}(G)) &= \pi_H^2 \frac{\delta}{n} \left[\frac{n-1+\delta n}{n^2(1-\pi_H)} - \frac{\delta}{1-\pi_H} \right] \\
&= \frac{\delta \pi_H^2 \left(2(n-1) - \delta n(n^2-2) \right)}{2n^3(1-\pi_H)},
\end{aligned}$$

which is positive for $\delta < \overline{\delta} := \frac{2(n-1)}{n(n^2-2)}$. This cutoff, $\overline{\delta}$ is decreasing in $n \geq 2$.

Therefore, for μ sufficiently close to 1, there exists a threshold $\overline{\delta}_n$ such that $\bar{c}^{\text{col}} > \bar{c}^{\text{ind}}$ if and only if $\delta < \overline{\delta}_n$, where $\lim_{\mu \to 1} \overline{\delta}_n = \overline{\delta} = \frac{2(n-1)}{n(n^2-2)}$.

□

B T-Period Memory and Quality Control

$T-$Period Memory Analysis and Proofs

In this section, we extend our analysis to a T-period memory for $T > 2$. With a $T-$period memory, a relevant history at period t is of the form $\mathbf{h}_t \in \mathcal{H}^{\text{ind}} := \{G, \varnothing, B\}^T$ for an individual brand and $\mathbf{h}_t \in \mathcal{H}^{\text{col}} := \{G, B\}^T$ for a collective brand. The history consists of outcomes produced in the previous T periods, $\mathbf{h}_t = h_{t-T} h_{t-T+1} \cdots h_{t-1}$. As periods pass, consumers' new history will consist of the most recent outcomes from $\mathbf{h}_t$ and new outcomes. Let us denote the n most recent outcomes by $\mathbf{h}_t^n = h_{t-n} \cdots h_{t-1}$ for any $1 \leq n \leq T$.

As in Section 4, we start by finding conditions under which the reputational equilibrium exists for an individual and a collective brand. Then, we compare the respective parameter regions to find where the equilibrium exists under a collective, but not under an individual brand. The analysis is similar to that in Section 4, so we omit details to avoid redundancy.

Individual brand

In a reputational equilibrium, a competent firm must find it optimal to invest after any history. To rule out profitable deviations, we consider the firm's investment decision at period t (also often referred as "today") given that the firm will invest whenever a consumer visits the firm. By investing, it can add $h_t = G$ to the history $\mathbf{h}_t$ with a greater probability, which will be remembered in the next T periods. $k+1$ periods after period t, consumers would have forgotten the $k+1$ oldest outcomes, and $k+1$ new outcomes are added to the relevant history

$$\mathbf{h}_{t+k+1} = \underbrace{h_{t-(T-k)} \cdots h_{t-1}}_{\text{old outcomes}=\mathbf{h}_t^{T-k}} \quad \underbrace{h_t h_{t+1} \cdots h_{t+k}}_{\text{new outcomes}=h_t \mathbf{h}_{t+k+1}^{k-1}} \quad .$$

The new outcomes are denoted by $h_t \mathbf{h}_{t+k+1}^{k-1}$, where h_t is the result of today's investment decision. Then, conditional on realizing the future outcomes $\mathbf{h}_{t+k+1}^{k-1}$, the benefit of investing in period t comes from a probabilistic improvement in the history from $\mathbf{h}_t^{T-k} B \mathbf{h}_{t+k+1}^{k-1}$ to $\mathbf{h}_t^{T-k} G \mathbf{h}_{t+k+1}^{k-1}$. This allows the firm to receive a higher price $p^{\text{ind}}(\mathbf{h}_t^{T-k} G \mathbf{h}_{t+k+1}^{k-1}) - p^{\text{ind}}(\mathbf{h}_t^{T-k} B \mathbf{h}_{t+k+1}^{k-1})$. The total expected benefit from a decision to invest today then is a sum of such price differences, weighted according to the probability of realizing $\mathbf{r}_{t+k+1}^{k}$ and accounting for an appropriate discounting.

So, we can compute the benefit of an investment for each history. Then, the reputational equilibrium exists if and only if the cost of investment is less than the minimum of benefits over all histories. We summarize this in the next lemma, which is a general statement of Lemma 1.

Lemma 3. *For an individual brand, there exists a constant $\overline{c}^{ind} > 0$ such that the reputatoinal equilibrium exists if and only if $c \leq \overline{c}^{ind}$ where*

$$\overline{c}^{ind} = \min_{\mathbf{h}_t^{T-1}} \overline{c}^{ind}(\boldsymbol{h}_t^{T-1}) := \frac{\delta \Delta \pi}{2} \cdot \sum_{k=0}^{T-1} \delta^k \Big(\sum_{\mathbf{f} \in \{G, \varnothing, B\}^k} \pi(\mathbf{f}) (p(\mathbf{h}_t^{T-k-1} G \mathbf{f}) - p(\mathbf{h}_t^{T-k-1} B \mathbf{f})) \Big), \quad (10)$$

where $\pi(\boldsymbol{f})$ denotes the probability distribution of future outcome $\boldsymbol{f}$ produced by a competenet firm in the reputational equilibrium.

Proof. As in Lemma 1, we obtain an expression for the cutoff in terms of price differences. Let $V(\mathbf{h}_t)$ be the expected payoff to the firm in equilibrium:

$$V^{\text{ind}}(\mathbf{h}_t) \equiv \tfrac{1}{2}(p(\mathbf{h}_t) - c) + \delta \left(\frac{\pi_H}{2} \cdot V^{\text{ind}}(\mathbf{h}_t^{T-1} G) + \frac{1-\pi_H}{2} \cdot V^{\text{ind}}(\mathbf{h}_t^{T-1} B) + \frac{1}{2} \cdot V^{\text{ind}}(\mathbf{h}_t^{T-1} \varnothing) \right).$$

As the consumer visits the firm with probability $\frac{1}{2}$, the firm's expected period-t profit is $\frac{1}{2}(p(\mathbf{h}_t) - c)$. The expected future payoff depends on the realized outcome in the current period. The firm produces outcomes G, B, $\varnothing$ with probabilities $\frac{\pi_H}{2}$, $\frac{1-\pi_H}{2}$, $\frac{1}{2}$, respectively.

Once the firm is visited, it should be optimal for the firm to invest always. Given a history

$\mathbf{h}_t$ and a consumer's visit, the expected payoff from following the equilibrium strategy is

$$p(\mathbf{h}_t) - c + \delta(\pi_H \cdot V^{\text{ind}}(\mathbf{h}_t^{T-1}G) + (1-\pi_H)\cdot V^{\text{ind}}(\mathbf{h}_t^{T-1}B)). \tag{11}$$

By deviating and not investing today, the firm expects to obtain the following payoff

$$p(\mathbf{h}_t) + \delta(\pi_L \cdot V^{\text{ind}}(\mathbf{h}_t^{T-1}G) + (1-\pi_L)\cdot V^{\text{ind}}(\mathbf{h}_t^{T-1}B)).$$

By investing in quality, the firm is able to produce a good outcome with a greater probability π_H, which improves the future payoffs. Then, the condition for the existence of the reputational equilibrium can be expressed as a cutoff-rule; the invest cost is always less than its benefit. So,

$$c \le \overline{c}^{\text{ind}} := \delta \cdot \Delta\pi \cdot \min_{\mathbf{h}_t^{T-1}\in\{G,\varnothing,B\}^{T-1}} \Delta V^{\text{ind}}(\mathbf{h}_t^{T-1}), \tag{12}$$

where $\Delta V^{\text{ind}}(\mathbf{h}_t^{T-1}) := V^{\text{ind}}(\mathbf{h}_t^{T-1}G) - V^{\text{ind}}(\mathbf{h}_t^{T-1}B)$. The firm then expects to receive a higher price in the next T periods due to the good outcome produced today. For this reason, $\Delta V(\mathbf{h}_t^{T-1})$ is a present-discounted weighted-sum of price premiums, as we saw in the analysis for two-period memory. The future payoff, conditional on producing a good outcome, is

$$\begin{aligned} V^{\text{ind}}(\mathbf{h}_t^{T-1}G) &= \underbrace{\frac{1}{2}\sum_{k=0}^{T-1}\delta^k \sum_{\mathbf{f}\in\{G,\varnothing,B\}^k} \pi(\mathbf{f})(\mathbf{f})(p(\mathbf{h}^{T-k-1}G\mathbf{f})-c)}_{\text{first } T \text{ periods}} + \underbrace{\frac{1}{2}\sum_{j=0}^{\infty}\delta^{T+j}\sum_{\mathbf{g}\in\{G,\varnothing,B\}^T}\Pr(\mathbf{g})(p(\mathbf{g})-c)}_{\text{after } T \text{ periods}}. \\ &= \frac{1}{2}\sum_{k=0}^{T-1}\delta^k\left(\sum_{i+j+l=k}\big(\frac{\pi_H}{2}\big)^i\big(\frac{1-\pi_H}{2}\big)^j\big(\frac{1}{2}\big)^l\left(\sum_{N_G(\mathbf{f})=i,N_B(\mathbf{f})=j} p(\mathbf{h}^{T-1-k}G\mathbf{f})\right)-c\right) \\ &+ \frac{1}{2}\delta^T\sum_{k=0}^{\infty}\delta^k\left(\sum_{i+j+l=T}\big(\frac{\pi_H}{2}\big)^i\big(\frac{1-\pi_H}{2}\big)^j\big(\frac{1}{2}\big)^l\left(\sum_{N_G(\mathbf{g})=i,N_B(\mathbf{g})=j} p(\mathbf{g})\right)-c\right). \end{aligned}$$

Given a history $\mathbf{h}_t^{T-1}G$, the relevant history k periods later becomes $\mathbf{h}^{T-k-1}G\mathbf{f}$. That is, consumers replace oldest k memories with a new memory realized throughout k periods, i.e., $\mathbf{f}\in\mathcal{H}^k$. Conditional on the realization of $\mathbf{f}$, the firm's per-period profit is $p(\mathbf{h}^{T-k-1}G\mathbf{f})-c$. This realization occurs with a probability denoted by $\Pr(\mathbf{f})$. Accouting for these probabilities and discounting, we obtain the first double sum in the equation. Once T periods have passed and consumers no longer remember the good outcome of the investment made in period t, the firm's relevant history can be any $T-$period history, $\mathbf{g}\in\mathcal{H}_T^{\text{ind}}$. So, we obtain the second double sum by weighting and discounting each per-period profit appropriately. The firm realized profit if and only if the consumer visits, and therefore we divide the whole expression by 2.

To compute $\pi(\mathbf{f})$, it is sufficient to count the number of good, bad and empty histories, as the order of each outcome is irrelevant for consumers' beliefs. Let $N_h(\mathbf{h}_t)$ for $h \in \{G, B, \varnothing\}$ and $\mathbf{h}_t \in \mathcal{H}_T^{\text{ind}}$ be the count of an outcome of type h in the T-period history $\mathbf{h}_t$. For example, $N_G(G\varnothing G) = 2$, $N_B(G\varnothing G) = 0$ and $N_\varnothing(G\varnothing G) = 1$. Suppose $N_G(\mathbf{f}) = i$, $N_B(\mathbf{f}) = j$, and $N_\varnothing(\mathbf{f}) = l$, respectively, such that $i + j + l = k$. Then, $\Pr(\mathbf{f}) = (\frac{\pi_H}{2})^i \cdot (\frac{1-\pi_H}{2})^j (\frac{1}{2})^l$. The next two lines in the equation are results of simply plugging in these probabilities.

Likewise, the future payoff to the firm if it produced a bad outcome would be

$$\begin{aligned} V^{\text{ind}}(\mathbf{h}_t^{T-1}B) \;=\; & \frac{1}{2}\sum_{k=0}^{T-1}\delta^k \left(\sum_{i+j+l=k} (\frac{\pi_H}{2})^i (\frac{1-\pi_H}{2})^j (\frac{1}{2})^l \left(\sum_{N_G(\mathbf{f})=i, N_B(\mathbf{f})=j} p(\mathbf{h}^{T-1-k}B\mathbf{f}) \right) - c \right) \\ & + \frac{1}{2}\delta^T \sum_{k=0}^{\infty} \delta^k \left(\sum_{i+j+l=T} (\frac{\pi_H}{2})^i (\frac{1-\pi_H}{2})^j (\frac{1}{2})^l \left(\sum_{N_G(\mathbf{g})=i, N_B(\mathbf{g})=j} p(\mathbf{g}) \right) - c \right). \end{aligned}$$

Therefore, subtracting the two gives

$$\begin{aligned} \Delta V^{\text{ind}}(\mathbf{h}_t^{T-1}) &= \tfrac{1}{2} \cdot \textstyle\sum_{k=0}^{T-1} \delta^k \sum_{\mathbf{f} \in \{G,\varnothing,B\}^k} \pi(\mathbf{f}) (p(\mathbf{h}^{T-k-1}G\mathbf{f}) - p(\mathbf{h}^{T-k-1}B\mathbf{f})) \\ &= \tfrac{1}{2} \cdot \textstyle\sum_{k=0}^{T-1} \delta^k \left(\sum_{i+j+l=k} (\frac{\pi_H}{2})^i (\frac{1-\pi_H}{2})^j (\frac{1}{2})^l \sum_{N_G(\mathbf{f})=i, N_B(\mathbf{f})=j} (p(\mathbf{h}^{T-1-k}G\mathbf{f}) - p(\mathbf{h}^{T-1-k}B\mathbf{f})) \right) \end{aligned}$$

Plugging this into (12) completes the proof. □

To obtain an explicit expression for $\bar{c}^{\text{ind}}$, we need to uncover the minimum operator by identifying the binding history for different parameter regions. As in the two-period memory case, we focus on two special signal structures: exclusive knowledge ($\pi_L = 0$) and quality control ($\pi_H = 1$). The former provides an environment where building an extremely high level of reputation is easy for a competent firm, as one good outcome completely reveals its type. Therefore, we can attain the minimum by choosing a history that has a lasting damage to the firm's incentives. This implies that any history $\mathbf{h}_t^{T-1}$ with $h_{t-1} = G$ does the job. As the most recent outcome in the history is good, consumers know perfectly the firm's type to be good until $t + T - 1$. This eliminates all the benefits to be realized until period $t + T - 1$. The only expression that survives in equation (10) is the very last period $(t + T)$ when $h_{t-1} = G$ will have been forgotten. As this benefit is discounted by δ^T, a longer history clearly hurts investment incentives for an individual brand.

Under the structure of quality control ($\pi_H = 1$), one bad outcome completely reveals a firm to be an incompetent type. Then, similarly, any history with $h_{t-1} = B$ attains the minimum because it puts a bad stamp on the brand for until period $t + T - 1$. Then, all benefits other than ones to be realized in the very last period $(t+T)$, again discounted by δ^T.

Therefore, $\lim_{\pi_L\to 0}\bar{c}^{\text{ind}} = \lim_{\pi_L\to 0}\bar{c}^{\text{ind}}(\mathbf{h}_t)$ where $h_{t-1}=G$, and $\lim_{\pi_H\to 1}\bar{c}^{\text{ind}} = \lim_{\pi_H\to 1}\bar{c}^{\text{ind}}(\mathbf{g}_t)$ where $g_{t-1}=B$. We state next lemma with characterization of the cutoff once we take limits for μ.

Lemma 4. *(i) In an the environment with exclusive knowledge ($\pi_L = 0$), a history in which the most recent outcome is G attains $\bar{c}^{ind}$. If μ is close to 1,*

$$\lim_{\mu\to 1}\lim_{\pi_L\to 0}\frac{\bar{c}^{ind}}{1-\mu} = \frac{\delta^T\pi_H^2}{2(1-\pi_H)} \tag{13}$$

(ii). In an environment with quality control ($\pi_H = 1$), a history in which the most recent outcome is B attains $\bar{c}^{ind}$. If μ is close to 0,

$$\lim_{\mu\to 0}\lim_{\pi_H\to 1}\frac{\bar{c}^{ind}}{\mu} = \frac{\delta^T(1-\pi_L)^2}{2^T\pi_L}\cdot(\frac{1+\pi_L}{\pi_L})^{T-1}. \tag{14}$$

Proof. First, the binding constraints are identified. Second, the cutoff-level is computed. As the exact cutoff level involves a minimum operator, we need to compare $\Delta V(\mathbf{h}_t^{T-1})$ for all $\mathbf{h}_t^{T-1}\in\{G,B,\varnothing\}$.

First, suppose $\pi_L = 0$, $\pi_H\in(0,1)$. This is the case of exclusive technology where a good outcome reveals the firm to be competent. So, $\Pr(\mathbf{h}) = 1$ if and only if $N_G(\mathbf{h})\geq 1$. Here, the price $p(\mathbf{h}) = \pi_H\cdot\Pr(\mathbf{h})$. So,

$$\begin{aligned} p(\mathbf{h}^{T-1-k}G\mathbf{f}) - p(\mathbf{h}^{T-1-k}B\mathbf{f}) &= \pi_H\cdot(\Pr(\mathbf{h}^{T-1-k}G\mathbf{f}) - \Pr(\mathbf{h}^{T-1-k}B\mathbf{f})) \\ &= \pi_H\cdot(1-\Pr(\mathbf{h}^{T-1-k}B\mathbf{f})). \end{aligned}$$

This vanishes if and only if $N_G(\mathbf{h}^{T-1-k}B\mathbf{f})\geq 1$, i.e. there is at least one good outcome in this history. To find a history that minimizes $\Delta V(\cdot)$, we want as many of the price difference as possible to vanish. For this purpose, it suffices to have $h_{t-1} = G$. Recall h_{t-1} is the outcome produced a period before the focal investment decision. So, the good outcome reveals the firm's competence until it is forgotten T periods later. So, with $h_{t-1} = G$, $p(\mathbf{h}^{T-1-k}G\mathbf{f}) - p(\mathbf{h}^{T-1-k}B\mathbf{f}) = 0$ for all $\mathbf{f}\in\{G,B,\varnothing\}^k$ for $0\leq k\leq T-2$. For $k = T-1$, h_{t-1} is forgotten and the relevant price premium is $p(G\mathbf{f}) - p(B\mathbf{f})$. So, for $h_{t-1} = G$,

$$\Delta V^{\text{ind}}(\mathbf{h}) \to_{\pi_L\to 0} \frac{1}{2}\cdot\delta^{T-1}\sum_{\mathbf{f}\in\mathcal{H}^{T-1}}\pi(\mathbf{f})(p(G\mathbf{f}) - p(B\mathbf{f}))$$

That is, all benefits other than the one realized in the last period vanish. And, this part is independent of $\mathbf{h}_t$, the history at the time of investment decision. Therefore, $h_{-1} = G$ indeed attains the minimum for $\Delta V(\cdot)$.

Clearly, $p(G\mathbf{f}) - p(B\mathbf{f}) = 0$ for any $N_G(\mathbf{f}) \geq 1$. Therefore, terms that survive in the equation above are $\mathbf{f}$ of length $T-1$ that only consists of B and/or $\varnothing$. Therefore,

$$\begin{aligned}
\lim_{\pi_L \to 0} \Delta V^{\text{ind}}(\mathbf{h}) &= \frac{\delta^{T-1}}{2} \cdot \left(\sum_{j=0}^{T-1} \binom{T-1}{j} (\frac{1-\pi_H}{2})^j (\frac{1}{2})^{T-1-j} \cdot \pi_H \left(\pi(GB^j \varnothing^{T-1-j}) - \pi(B^{j+1} \varnothing^{T-1-j}) \right) \right) \\
&= \frac{\delta^{T-1}}{2} \cdot \left(\sum_{j=0}^{T-1} \binom{T-1}{j} (\frac{1-\pi_H}{2})^j (\frac{1}{2})^{T-1-j} \cdot \pi_H \left(1 - \frac{\mu(1-\pi_H)^{j+1}}{\mu(1-\pi_H)^{j+1} + 1 - \mu} \right) \right) \\
&= \frac{\pi_H(1-\mu)}{2^T} \cdot \delta^{T-1} \left(\sum_{j=0}^{T-1} \binom{T-1}{j} \frac{(1-\pi_H)^j}{\mu(1-\pi_H)^{j+1} + (1-\mu)} \right).
\end{aligned}$$

The first equality holds because $\pi(GB^j \varnothing^{T-1-j}) = 1$ because a good history causes a full revelation, and $\pi(B^{j+1}\varnothing^{T-1-j}) = \frac{\mu(1-\pi_H)^{j+1}}{\mu(1-\pi_H)^{j+1}+1-\mu}$. Simply plugging into (12) proves the lemma for $\pi_L = 0$ and $\pi_H \in (0,1)$. In particular, $\lim_{\mu \to 1} \lim_{\pi_L \to 0} \frac{\Delta V^{\text{ind}}(\mathbf{h})}{1-\mu} = \frac{\pi_H}{2(1-\pi_H)} \cdot \delta^{T-1}$.

Now, consider the case where $\pi_H = 1$ and $\pi_L \in (0,1)$. Here, a bad outcome is revealing of a firm's incompetence. Therefore, $\mu(\mathbf{h}) = 0$ if and only if $N_B(\mathbf{h}) \geq 1$, and $p(\mathbf{h}) = \pi_L$. We omit details for this case, as it is very similar to the previous case.

From Equation (13), $h_{t-1} = B$ attains the minimum for $\Delta V^{\text{ind}}(\cdot)$. Then, all price premiums other than the ones to be realized in the last period, $t+T-1$, vanish. Therefore,

$$\begin{aligned}
\Delta V^{\text{ind}}(\mathbf{h}) &\to_{\pi_H \to 1} \frac{1}{2} \cdot \delta^{T-1} \sum_{\mathbf{f} \in \mathcal{H}^{T-1}} \pi(\mathbf{f})(p(G\mathbf{f}) - p(B\mathbf{f})) \\
&= \frac{\delta^{T-1}(1-\pi_L)}{2} \left(\sum_{j=0}^{T-1} \binom{T-1}{j} (\frac{1}{2})^j (\frac{1}{2})^{T-1-j} \left(\pi(G^{j+1}\varnothing^{T-1-j}) - \pi(BG^j\varnothing^{T-1-j}) \right) \right) \\
&= \frac{\delta^{T-1}(1-\pi_L)\mu}{2^T} \left(\sum_{j=0}^{T-1} \binom{T-1}{j} \frac{1}{\mu + (1-\mu)\pi_L^{j+1}} \right).
\end{aligned}$$

Plugging this into (12) completes the proof. In particular,

$$\lim_{\mu \to 0} \lim_{\pi_H \to 1} \frac{\Delta V^{\text{ind}}(\mathbf{h})}{\mu} = \frac{\delta^{T-1}(1-\pi_L)}{2^T \pi_L} \cdot (\frac{1+\pi_L}{\pi_L})^{T-1}. \tag{15}$$

□

Collective brand

A longer memory also allows a collective brand to reach a higher level of reputation by producing good outcomes, which reduces incentives for firms in the group to further exert costly investments. However, as we saw in the analysis of the main model with a two-period memory, consumers' limited observability for a collective brand alleviates this problem; as consumers cannot observe history at firm-level, they can never learn perfectly about the types of two firms in the group. Therefore, a competent firm can always improve the brand reputation by investing in quality.

The relevant history for a collective brand with $T-$period memory is $\mathbf{h}_t \in \mathcal{H}_T^{\text{col}} = \{G, B\}^T$. The next lemma establishes the necessary and sufficient condition for the existence of reputational equilibrium. Let $\theta \in \{C, I\}$ denote the other firm's type. $\pi(\mathbf{f}; \theta)$ for $\mathbf{f} \in \{G, B\}^k$ and $\theta \in \{C, I\}$ with $0 \leq k \leq T$ is the the probability that the brand produces a sequence of outcomes $\mathbf{f}$ in k periods if a competent firm always invests.

Lemma 5. *For a competent firm within a collective brand, there exists a constant $\overline{c}^{col} > 0$ such that the reputational equilibrium exists if and only if $c \leq \overline{c}^{col}$ where*

$$\overline{c}^{col} = \min_{\boldsymbol{h}_t^{T-1}, \theta} \overline{c}^{col}(\boldsymbol{h}_t^{T-1}, \theta) := \frac{\delta \Delta \pi}{2} \cdot \sum_{k=0}^{T-1} \delta^k \Big(\sum_{\mathbf{f} \in \{G,B\}^k} \pi(\mathbf{f}; \theta)(p(\mathbf{h}_t^{T-k-1} G \mathbf{f}) - p(\mathbf{h}_t^{T-k-1} B \mathbf{f})) \Big), \tag{16}$$

where $\mathbf{h}_t^{T-1} \in \{G, B\}^{T-1}$ and $\theta \in \{C, I\}$.

Proof. As this lemma is a straightforward generalization of Lemma 2, we omit many details. Also, we adopt notation from the proof for Lemma 3. Let $V_\theta^{\text{col}}(\mathbf{h}_t)$ denote the payoff to a competent firm of a collective brand before the customer's visit.

$$V_\theta^{\text{col}}(\mathbf{h}_t) \equiv \underbrace{\frac{1}{2}(p(\mathbf{h}_t) - c)}_{\text{current period profit}} + \delta \left(\underbrace{\frac{\pi_H + \pi(\theta)}{2} \cdot V_\theta^{\text{col}}(\mathbf{h}_t^{T-1} G) + (1 - \frac{\pi_H + \pi(\theta)}{2}) \cdot V_\theta^{\text{col}}(\mathbf{h}_t^{T-1} B)}_{\text{continuation payoff}} \right).$$

In the current period the firm makes $p(\mathbf{h}_t) - c$ if visited and 0 otherwise. In the next period, the brand will face a history $\mathbf{h}_t^{T-1} G$ or $\mathbf{h}_t^{T-1} B$ depending on today's investment outcome, which also depends on the type of the other firm. So, on average, the firm produces a G with a probability $\frac{\pi_H + \pi(\theta)}{2}$ and a B otherwise.

Once the firm is visited, it should be optimal for the firm to invest always. After a history $\mathbf{h}_t$, by following the equilibrium strategy, the firm expects to receive

$$p(\mathbf{h}_t) - c + \delta(\pi_H \cdot V_\theta^{\text{col}}(\mathbf{h}_t^{T-1} G) + (1 - \pi_H) \cdot V_\theta^{\text{col}}(\mathbf{h}_t^{T-1} B))$$

The firm's expected payoff from a deviation is

$$p(\mathbf{h}_t) + \delta(\pi_L \cdot V_\theta^{\text{col}}(\mathbf{h}_t^{T-1}G) + (1-\pi_L) \cdot V_\theta^{\text{col}}(\mathbf{h}_t^{T-1}B)).$$

This is equivalent to

$$c \leq \overline{c}^{\text{col}} := \delta \cdot \Delta\pi \cdot \min_{\mathbf{h}_t^{T-1} \in \{G,B\}^{T-1}} \Delta V_\theta^{\text{col}}(\mathbf{h}_t^{T-1}), \tag{17}$$

where $\Delta V_\theta^{\text{col}}(\mathbf{h}_t^{T-1}) := V_\theta^{\text{col}}(\mathbf{h}_t^{T-1}G) - V_\theta^{\text{col}}(\mathbf{h}_t^{T-1}B)$.

The future payoff, conditional on producing an outcome of either G or B, is

$$V_\theta^{\text{col}}(\mathbf{h}_t^{T-1}G) = \underbrace{\frac{1}{2}\sum_{k=0}^{T-1}\delta^k \sum_{\mathbf{f}\in\mathcal{H}^k}\pi(\mathbf{f};\theta)(p(\mathbf{h}^{T-k-1}G\mathbf{f}) - c)}_{\text{First } T \text{ Periods}} + \underbrace{\frac{1}{2}\sum_{j=0}^{\infty}\delta^{T+j}\sum_{\mathbf{g}\in\mathcal{H}^T}\pi(\mathbf{g};\theta)(p(\mathbf{g}) - c)}_{\text{After } T \text{ Periods}},$$

$$V_\theta^{\text{col}}(\mathbf{h}_t^{T-1}B) = \underbrace{\frac{1}{2}\sum_{k=0}^{T-1}\delta^k \sum_{\mathbf{f}\in\mathcal{H}^k}\pi(\mathbf{f};\theta)(p(\mathbf{h}^{T-k-1}B\mathbf{f}) - c)}_{\text{First } T \text{ Periods}} + \underbrace{\frac{1}{2}\sum_{j=0}^{\infty}\delta^{T+j}\sum_{\mathbf{g}\in\mathcal{H}^T}\pi(\mathbf{g};\theta)(p(\mathbf{g}) - c)}_{\text{After } T \text{ Periods}}$$

In each period, the brand produces a G with a probability $\frac{\pi_H+\pi(\theta)}{2}$ and a B with the complementary probility. Therefore, for any $\mathbf{h}_t \in \mathcal{H}^{\text{col}}$, if $N_G(\mathbf{h}_t) = i$ and $N_B(\mathbf{h}_t) = j = t-i$, $\pi(\mathbf{f};\theta) = (\frac{\pi_H+\pi(\theta)}{2})^i(1-\frac{\pi_H+\pi(\theta)}{2})^j$.

Therefore, subtracting the two gives

$$\begin{aligned}\Delta V_\theta^{\text{col}}(\mathbf{h}_t^{T-1}) &= \frac{1}{2}\cdot\sum_{k=0}^{T-1}\delta^k\sum_{\mathbf{f}\in\mathcal{H}^k}\pi(\mathbf{f};\theta)(p(\mathbf{h}^{T-k-1}G\mathbf{f}) - p(\mathbf{h}^{T-k-1}B\mathbf{f})) \qquad (18)\\ &= \frac{1}{2}\cdot\sum_{k=0}^{T-1}\delta^k\left(\sum_{i+j=k}(\frac{\pi_H+\pi(\theta)}{2})^i(1-\frac{\pi_H+\pi(\theta)}{2})^j\left(\sum_{N_G(\mathbf{f})=i}(p(\mathbf{h}^{T-1-k}G\mathbf{f}) - p(\mathbf{h}^{T-1-k}B\mathbf{f}))\right)\right)\end{aligned}$$

Plugging this into (17) completes the proof. □

This lemma generalizes lemma 2. The cutoff now depends on the type of the other firm, as it affects realization of future outcomes $\mathbf{f}$ through $\pi(\mathbf{f};\theta)$. Also, prices here are different from those in the individual brand because conditional on a history, posterior beliefs are different.

First, in the exclusive knowledge case, $\pi_L = 0$ and μ close to 1. Then, a good outcome is informative. However, the informativeness of each additional good outcome must be decreasing. For example, having one good outcome compared to none is quite desirable, as it reveals the existence of at least one competent firm. But, having a fifth good outcome in the history in addition to existing four is not as valuable, as consumers already believe with a high probability that both firms are competent. So, in this parameter region, the binding constraint would be provided by an environment that produces as many good outcomes as possible. Naturally, $\mathbf{h}_t^{T-1} = G^{T-1}$ and $\theta = C$ would do the job.

Second, in quality control, $\pi_H = 1$ and μ close to 0. Then, although a bad outcome is informative, it's informativeness decreases as there are more bad outcomes in the history. So, the binding condition would be provided by $\mathbf{h}_t^{T-1} = B^{T-1}$ and $\theta = I$, as together they produce as many bad outcomes as possible in the brand's history.

Then, we can compute the cutoff levels explicitly:

Lemma 6. *(i) Under the environment of exclusive technology ($\pi_L = 0$), if μ is close to 1, $\bar{c}^{col} = \bar{c}^{col}(G^{T-1}, C)$ and*

$$\lim_{\mu \to 1} \lim_{\pi_L \to 0} \frac{\bar{c}^{col}}{1-\mu} = \frac{\delta \pi_H^2}{2^{T+1}(1-\pi_H)} \cdot \frac{1-(2\delta)^T}{1-2\delta} \tag{19}$$

(ii) Under the quality control ($\pi_H = 1$), if μ is close to 0, $\bar{c}^{col} = \bar{c}^{col}(B^{T-1}, I)$ and

$$\lim_{\mu \to 0} \lim_{\pi_H \to 1} \frac{\bar{c}^{col}}{\mu} = \frac{\delta(1-\pi_L)^2}{2^{T+1}\pi_L} \cdot \frac{1-(\frac{\delta}{2}\frac{1+3\pi_L}{\pi_L})^T}{1-\frac{\delta}{2}\frac{1+3\pi_L}{\pi_L}} \tag{20}$$

Proof. The exact cutoff levels in lemma 5 is a discounted sum of price premiums over T periods. It is not feasible to obtain an explicit expression for general parameter regions. We find it useful to understand posterior beliefs denoted by $\eta(\cdot)$. Facing a collective brand, consumers update beliefs over types of the brand, $s \in \{CC, CI, IC, II\}$, and use this to compute the probability of visiting a competent firm: $\Pr^{\text{col}}(\mathbf{h}_t) = \eta_{CC}(\mathbf{h}_t) + \frac{1}{2}(\eta_{CI}(\mathbf{h}_t) + \eta_{IC}(\mathbf{h}_t))$. So, $\Pr^{\text{col}}(\mathbf{h}_t)$, if $N_G(\mathbf{h}_t) = i$, is

$$\Pr^{\text{col}}(\mathbf{h}_t) = \frac{\mu^2 \cdot \pi_H^i(1-\pi_H)^{T-i} + \mu(1-\mu)\cdot(\frac{\pi_H+\pi_L}{2})^i(1-\frac{\pi_H+\pi_L}{2})^{T-i}}{\mu^2 \cdot \pi_H^i(1-\pi_H)^{T-i} + 2\mu(1-\mu)\cdot(\frac{\pi_H+\pi_L}{2})^i(1-\frac{\pi_H+\pi_L}{2})^{T-i} + (1-\mu)^2\cdot\pi_L^i(1-\pi_L)^{T-i}} \tag{21}$$

It is infeasible to obtain an explicit expression for $\Delta V_\theta(\cdot)$, not to mention the overall cutoff, $\bar{c}^{\text{col}}$. As we did in previous analyses, we i) focus on two signal structures ($\pi_L = 0$ vs. $\pi_H = 1$), ii) identify the binding history and the brand type, and iii) obtain the cutoff level.

First, consider the case $\pi_L = 0$. Then, after a history $\mathbf{h}_t$, the consumer pays $p(\mathbf{h}_t) = \Pr^{\text{col}}(\mathbf{h}_t) \cdot \pi_H$. The reputational benefit realized in each period is the price difference made available by one more good outcome in the history, and thus is of a form $p(\mathbf{h}^{T-1-k}G\mathbf{f}) - p(\mathbf{h}^{T-1-k}B\mathbf{f})$, where $N_G(\mathbf{h}^{T-1-k}G\mathbf{f}) = N_G(\mathbf{h}^{T-1-k}B\mathbf{f}) + 1$. And, here we claim that this difference is decreasing in i for a large enough μ. That is, when $\pi_L = 0$ and μ is large, the price premium reduces as the number of good outcomes becomes large. If this were true, $\mathbf{h}_t^{T-1} = G^{T-1}$ and $\theta = C$ would provide the minimum for $\Delta V_\theta(\mathbf{h}_t^{T-1})$, as these two conditions both places the brand under histories with more good outcomes. We formally state this and prove:

Claim 1. Suppose $\pi_L = 0$ and μ is close to 1. Let $\boldsymbol{r}_1, \boldsymbol{r}_2 \in \mathcal{H}_T^{col}$ such that $N_G(\boldsymbol{r}_1) = i+1$ and $N_G(\boldsymbol{r}_2) = i$. Then, $p(\mathbf{r}_1) - p(\mathbf{r}_2)$ is decreasing in i. So, $\boldsymbol{h}_t^{T-1} = G^{T-1}$ and $\theta = C$ attains the minimum for $\Delta V_\theta(\boldsymbol{h}_t^{T-1})$, and hence are the binding condition for the cutoff level, $\bar{c}^{col}$.

The intuition is the following. As long as there is a good outcome in the history, consumers believe the brand has either one or two competent firms. But, as they see more good outcomes, they become more convinced that both firms are competent. As more good outcomes resolve consumers' uncertainty, the price difference becomes small. Mathematically, from equation (21),

$$\begin{aligned}
\lim_{\pi_L \to 0} (\mathrm{Pr}^{\mathrm{col}}(\mathbf{r}_1) - \mathrm{Pr}^{\mathrm{col}}(\mathbf{r}_2)) &= \frac{\mu^2 \cdot \pi_H^{i+1}(1-\pi_H)^{T-i-1} + \mu(1-\mu)\cdot(\frac{\pi_H}{2})^{i+1}(1-\frac{\pi_H}{2})^{T-i-1}}{\mu^2 \cdot \pi_H^{i+1}(1-\pi_H)^{T-i-1} + 2\mu(1-\mu)\cdot(\frac{\pi_H}{2})^{i+1}(1-\frac{\pi_H}{2})^{T-i-1}} \\
&- \frac{\mu^2 \cdot \pi_H^{i}(1-\pi_H)^{T-i} + \mu(1-\mu)\cdot(\frac{\pi_H}{2})^{i}(1-\frac{\pi_H}{2})^{T-i}}{\mu^2 \cdot \pi_H^{i}(1-\pi_H)^{T-i} + 2\mu(1-\mu)\cdot(\frac{\pi_H}{2})^{i}(1-\frac{\pi_H}{2})^{T-i}} \\
&= \frac{\mu(1-\mu)\cdot(\frac{\pi_H}{2})^{i}(1-\frac{\pi_H}{2})^{T-i}}{\mu^2 \cdot \pi_H^{i}(1-\pi_H)^{T-i} + 2\mu(1-\mu)\cdot(\frac{\pi_H}{2})^{i}(1-\frac{\pi_H}{2})^{T-i}} \\
&- \frac{\mu(1-\mu)\cdot(\frac{\pi_H}{2})^{i+1}(1-\frac{\pi_H}{2})^{T-i-1}}{\mu^2 \cdot \pi_H^{i+1}(1-\pi_H)^{T-i-1} + 2\mu(1-\mu)\cdot(\frac{\pi_H}{2})^{i+1}(1-\frac{\pi_H}{2})^{T-i-1}}
\end{aligned}$$

Then, taking $\frac{\mathrm{Pr}^{\mathrm{col}}(\mathbf{r}_1) - \mathrm{Pr}^{\mathrm{col}}(\mathbf{r}_2)}{1-\mu}$ to a limit as $\mu \to 1$,

$$\begin{aligned}
\lim_{\mu \to 1} \lim_{\pi_L \to 0} \frac{\mathrm{Pr}^{\mathrm{col}}(\mathbf{r}_1) - \mathrm{Pr}^{\mathrm{col}}(\mathbf{r}_2)}{1-\mu} &= \frac{(\frac{\pi_H}{2})^{i}(1-\frac{\pi_H}{2})^{T-i}}{\pi_H^{i}(1-\pi_H)^{T-i}} - \frac{(\frac{\pi_H}{2})^{i+1}(1-\frac{\pi_H}{2})^{T-i-1}}{\pi_H^{i+1}(1-\pi_H)^{T-i-1}} \\
&= \frac{1}{(1-\pi_H)2^{i+1}}(\frac{1-\frac{\pi_H}{2}}{1-\pi_H})^{T-i-1},
\end{aligned}$$

which is clearly decreasing in i. Therefore, for any positive integer T, there is a $\bar{\mu}$ close enough to 1 so that, for any $\mu > \overline{\mu}$, the difference in beliefs (and thus prices) is decreasing in i, the number of good outcomes in the history. This completes the proof for the claim.

Then, we plug in $\mathbf{h}^{T-1} = G^{T-1}$ and $\theta = C$ to compute:

$$\begin{aligned}
\lim_{\mu\to 1}\lim_{\pi_L\to 0}\frac{\Delta V_C^{\mathrm{col}}(G^{T-1})}{1-\mu} &= \frac{\pi_H}{2}\cdot\sum_{k=0}^{T-1}\delta^k\left(\sum_{i+j=k}\pi_H^i(1-\pi_H)^j\left(\sum_{N_G(\mathbf{f})=i}(\Pr^{\mathrm{col}}(\mathbf{h}^{T-1-k}G\mathbf{f})-\Pr^{\mathrm{col}}(\mathbf{h}^{T-1-k}B\mathbf{f}))\right)\right)\\
&= \frac{\pi_H}{2(1-\pi_H)}\cdot\sum_{k=0}^{T-1}\delta^k\left(\sum_{i+j=k}\pi_H^i(1-\pi_H)^j\binom{k}{i}\frac{1}{2^{T-k+i}}(\frac{1-\frac{\pi_H}{2}}{1-\pi_H})^j\right)\\
&= \frac{\pi_H}{2^{T+1}(1-\pi_H)}\cdot\sum_{k=0}^{T-1}(2\delta)^k\\
&= \frac{\pi_H}{2^{T+1}(1-\pi_H)}\cdot\frac{1-(2\delta)^T}{1-2\delta}
\end{aligned}$$

Next, we consider the case $\pi_H = 1$. Then, the price consumer pays after a history $\mathbf{h}_t$ is $p(\mathbf{h}_t) = \Pr^{\mathrm{col}}(\mathbf{h}_t) + (1-\Pr^{\mathrm{col}}(\mathbf{h}_t))\pi_L$. In this setting, a bad outcome is very informative, as it reveals existence of an incompetent firm in the brand. And, intuitively as there are more bad outcomes in the history, informativeness of each bad outcome decrease. Therefore, the price premium to be realized k period after the focal investment decision conditional on the new outcomes $\mathbf{f}$ is $p(\mathbf{h}^{T-1-k}G\mathbf{f}) - p(\mathbf{h}^{T-1-k}B\mathbf{f})$, and this decreases in i, where $i = N_G(\mathbf{h}^{T-1-k}B\mathbf{f})$. We state it formally in the next claim.

Claim 2. Suppose $\pi_H = 1$ and μ is close to 0. And let $N_G(\mathbf{h}^{T-1-k}B\mathbf{f}) = i$. Then, $p(\mathbf{h}^{T-1-k}G\mathbf{f}) - p(\mathbf{h}^{T-1-k}B\mathbf{f})$ is increasing in i. Therefore, $\boldsymbol{h}_t^{T-1} = B^{T-1}$ and $\theta = I$ attains the minimum for $\Delta V_\theta(\boldsymbol{h}_t^{T-1})$, and hence provide the binding condition for the cutoff level, $\bar{c}^{col}$.

From equation (21),

$$\begin{aligned}
\lim_{\pi_H\to 1}\Pr^{\mathrm{col}}(G^T) &= \frac{\mu^2+\mu(1-\mu)\cdot(\frac{1+\pi_L}{2})^T}{\mu^2+2\mu(1-\mu)\cdot(\frac{1+\pi_L}{2})^T+(1-\mu)^2\cdot\pi_L^T}\\
\lim_{\pi_H\to 1}\Pr^{\mathrm{col}}(\mathbf{h}) &= \frac{\mu(1-\mu)\cdot(\frac{1+\pi_L}{2})^i(\frac{1-\pi_L}{2})^{T-i}}{2\mu(1-\mu)\cdot(\frac{1+\pi_L}{2})^i(\frac{1-\pi_L}{2})^{T-i}+(1-\mu)^2\cdot\pi_L^i(1-\pi_L)^{T-i}}
\end{aligned}$$

Then, $\lim_{\pi_H\to 1}(\Pr^{\mathrm{col}}(\mathbf{r}_1) - \Pr^{\mathrm{col}}(\mathbf{r}_2)) =$

$$\begin{aligned}
&\frac{\mu(1-\mu)\cdot(\frac{1+\pi_L}{2})^{i+1}(\frac{1-\pi_L}{2})^{T-i-1}}{\mu(1-\mu)\cdot(\frac{1+\pi_L}{2})^{i+1}(\frac{1-\pi_L}{2})^{T-i-1}+(1-\mu)^2\cdot\pi_L^{i+1}(1-\pi_L)^{T-i-1}}\\
-\quad&\frac{\mu(1-\mu)\cdot(\frac{1+\pi_L}{2})^i(\frac{1-\pi_L}{2})^{T-i}}{\mu(1-\mu)\cdot(\frac{1+\pi_L}{2})^i(\frac{1-\pi_L}{2})^{T-i}+(1-\mu)^2\cdot\pi_L^i(1-\pi_L)^{T-i}}
\end{aligned}$$

Then, taking $\frac{\Pr^{\text{col}}(\mathbf{r}_1)-\Pr^{\text{col}}(\mathbf{r}_2)}{\mu}$ to a limit as $\mu \to 0$,

$$\begin{aligned}\lim_{\mu\to 0}\lim_{\pi_H\to 1}\frac{\Pr^{\text{col}}(\mathbf{r}_1)-\Pr^{\text{col}}(\mathbf{r}_2)}{\mu} &= (\frac{1+\pi_L}{2\pi_L})^{i+1}\frac{1}{2^{T-i-1}}-(\frac{1+\pi_L}{2\pi_L})^{i}\frac{1}{2^{T-i}} \\ &= \frac{1}{2^T}\frac{(1+\pi_L)^i}{\pi_L^{i+1}}.\end{aligned}$$

This is clearly increasing in i. Therefore, there is a $\bar{\mu}_{\pi_H=1}$ close enough to 0 so that the difference in beliefs (and thus prices) is increasing in i, the number of good outcomes in the history. This completes the proof for the claim.

Then, we plug in $\mathbf{h}^{T-1}=B^{T-1}$ and $\theta=I$ to compute:

$$\begin{aligned}\lim_{\mu\to 0}\lim_{\pi_H\to 1}\frac{\Delta V_I^{\text{col}}(B^{T-1})}{\mu} &= \frac{1-\pi_L}{2}\cdot\sum_{k=0}^{T-1}\delta^k\left(\sum_{i+j=k}(\frac{1+\pi_L}{2})^i(\frac{1-\pi_L}{2})^j\left(\sum_{N_G(\mathbf{f})=i}(\frac{1}{2^T}\frac{(1+\pi_L)^i}{\pi_L^{i+1}})\right)\right) \\ &= \frac{1-\pi_L}{2^{T+1}}\cdot\sum_{k=0}^{T-1}\delta^k\left(\sum_{i+j=k}(\frac{1+\pi_L}{2})^i(\frac{1-\pi_L}{2})^j\binom{k}{i}\frac{(1+\pi_L)^i}{\pi_L^{i+1}}\right) \\ &= \frac{1-\pi_L}{2^{T+1}\pi_L}\cdot\sum_{k=0}^{T-1}\frac{\delta^k}{2^k}\left(\frac{1+3\pi_L}{\pi_L}\right)^k \\ &= \frac{1-\pi_L}{2^{T+1}\pi_L}\cdot\frac{1-\frac{\delta^T}{2^T}(\frac{1+3\pi_L}{\pi_L})^T}{1-\frac{\delta}{2}(\frac{1+3\pi_L}{\pi_L})} \qquad (22)\end{aligned}$$

□

Even in the limits, benefits of investment for a collective brand do not vanish, and the cutoff turns out to be a sum of what turns out to be a finite geometric sequence. Unlike the cutoff for an individual brand, the cutoff is not discounted by δ^T, so it decreases in T as a much slower rate. This highlights the advantage of collective brands over individual ones.

Comparing Individual and Collective Brands

It remains to prove the statement in Proposition 6, in particular the conditions under which $\bar{c}^{\text{col}}$ is greater than $\bar{c}^{\text{ind}}$. We compare the cutoff levels obtained in euqations (13) and (19), and (14) and (20).

Proof. [Proposition 6] For the good news case with $\pi_L=0$, we compare the cutoff levels we obtained by taking limit of μ to 1. $\bar{c}^{\text{col}} \geq \bar{c}^{\text{ind}}$ in this region if $\lim_{\mu\to 1}\lim_{\pi_L\to 0}\frac{\Delta V_C^{\text{col}}(G^{T-1})}{1-\mu} >$

$\lim_{\mu \to 1} \lim_{\pi_L \to 0} \frac{\Delta V^{\text{ind}}(G^{T-1})}{1-\mu}$.

$$\begin{aligned} & \frac{\pi_H}{2^{T+1}(1-\pi_H)} \cdot \frac{1-(2\delta)^T}{1-2\delta} > \frac{\pi_H}{2(1-\pi_H)} \cdot \delta^{T-1} \\ \Leftrightarrow \quad & \delta \cdot \frac{1-(2\delta)^T}{1-2\delta} > (2\delta)^T. \end{aligned} \tag{23}$$

If $\delta < \frac{1}{2}$, this condition is equivalent to $\frac{\delta}{1-2\delta} > \frac{(2\delta)^T}{1-(2\delta)^T}$. This holds true for every $T \geq 2$. This is because when $T = 2$, $\frac{\delta}{1-2\delta} > \frac{(2\delta)^2}{1-(2\delta)^2}$ if and only if $\delta < 1/2$. Also, the right-hand side is decreasing in T for $\delta < 1/2$.

If $\delta > \frac{1}{2}$, the condition is equivalent to $\frac{(2\delta)^T-1}{(2\delta)^T} > 2 \cdot \frac{2\delta-1}{2\delta}$. Putting $x = 2\delta$, this condition can be re-written as $\frac{x^T-1}{x^T} > 2 \cdot \frac{x-1}{x}$, which is equal to $(2-x)x^{T-1} > 1$. Because $x = 2\delta \in (1,2)$, the left-hand side is increasing in T. In particular, as T approaches ∞, the left-hand side diverges to ∞, and therefore the condition holds for all $\delta(1/2, 1)$. To understand the condition in terms of δ, we plug back in $x = 2\delta$, which yields $(2\delta)^{T-1} > \frac{1}{2(1-\delta)}$. By taking log on both sides, this condition holds if and only if $(T-1)\log(2\delta) > \log \frac{1}{2(1-\delta)}$. If $\delta \searrow \frac{1}{2}$, the condition holds for any $T \geq 3$. On the other hand, as $\delta \nearrow 1$, the right-hand side diverges, whereas the left-hand side converges. Therefore, for any fixed $T \geq 3$, the condition holds for all $\delta < \overline{\delta}(T)$. And, as shown above, as $T \to \infty$, $\overline{\delta}(T) \to 1$ so that collective brand sustains the reputational equilibrium better than individual brands can for all $\delta \in [0,1]$.

For the case with $\pi_H = 1$, we compare $\lim_{\mu \to 0} \lim_{\pi_H \to 1} \frac{\Delta V^{\text{ind}}(B^{T-1})}{\mu}$ from equation (15) and $\lim_{\mu \to 0} \lim_{\pi_H \to 1} \frac{\Delta V_I^{\text{col}}(B^{T-1})}{\mu}$ (22), and a collective brand sustains the reputational equilibrium better if and only if

$$\begin{aligned} & \frac{1-\pi_L}{2^{T+1} \cdot \pi_L} \cdot \frac{1-\frac{\delta^T}{2^T}(\frac{1+3\pi_L}{\pi_L})^T}{1-\frac{\delta}{2}(\frac{1+3\pi_L}{\pi_L})} > \frac{\delta^{T-1}(1-\pi_L)}{2^T \cdot \pi_L} \cdot (\frac{1+\pi_L}{\pi_L})^{T-1} \\ \Leftrightarrow \quad & \frac{1}{2} \cdot \frac{1-\frac{\delta^T}{2^T}(\frac{1+3\pi_L}{\pi_L})^T}{1-\frac{\delta}{2}(\frac{1+3\pi_L}{\pi_L})} > (\delta \cdot \frac{1+\pi_L}{\pi_L})^{T-1} \end{aligned} \tag{24}$$

If $T = 2$, this condition holds if and only if $\delta < \frac{2\pi_L}{3+\pi_L}$, which coincides with the condition identified in the proof of Proposition 8.

Because the left-hand side is always increasing in T, (24) is more likely to hold if $\frac{\delta(1+\pi_L)}{\pi_L} \leq 1$. Otherwise, if $\frac{\delta(1+\pi_L)}{\pi_L} > 1$, the right-hand side diverges as T goes to infinity. So, in order for the condition to hold, the left-hand side must diverge at a faster rate. The left-hand side converges if and only if $\delta < \frac{2\pi_L}{1+3\pi_L}$. So, if $\frac{\pi_L}{1+\pi_L} < \delta < \frac{2\pi_L}{1+3\pi_L}$, the condition holds only for a small enough T.

If $\delta > \frac{2\pi_L}{1+3\pi_L}$, we can show that the condition cannot hold for T too large. The condition above is equivalent to $(\frac{\delta(1+3\pi_L)}{2\pi_L})^T - 1 > 2 \cdot \left(\frac{\delta}{2}(\frac{1+3\pi_L}{\pi_L})\right) \cdot \left(\frac{\delta(1+\pi_L)}{\pi_L}\right)^{T-1}$. Then, for a very large T, this condition does not hold because $\frac{\delta(1+3\pi_L)}{2\pi_L} < \frac{\delta(1+\pi_L)}{\pi_L}$.

□

Proofs for the Quality Control Case ($\pi_H = 1$)

This section formally analyses the statements made in the article for the quality control case, i.e. when $\pi_H = 1$. In this environment, a competent firm always produces a good outcome if it exerts investment efforts. An incompetent firm (and a competent firm that does not invest) produces a good outcome with probability $\pi_L \in (0, 1)$. Thus, upon producing a bad outcome, the market concludes that the firm is incompetent with certainty.

Proposition 8. *Suppose $\pi_H = 1$ so that a bad outcome reveals a firm's incompetence. For μ sufficiently close to 0, there exists $\overline{\delta}$ such that $\overline{c}^{col} > \overline{c}^{ind}$ if and only if $\delta < \overline{\delta}$. As μ approaches 0, $\overline{\delta}$ converges to $\frac{2\pi_L}{3+\pi_L}$. If μ is sufficiently close to 1, $\overline{c}^{col} \leq \overline{c}^{ind}$ for all $\delta \in [0, 1]$.*

Proof. For $\pi_H = 1$, it follows from Lemma 1 that $\overline{c}^{\text{ind}} = \overline{c}^{\text{ind}}(B)$. Also, from Lemma 2, $\overline{c}^{\text{col}} = \overline{c}^{\text{col}}(B; I)$ given $\pi_H = 1$ for all $\mu \in [0, 1]$. With an individual brand, $\lim_{\pi_H \to 1} \overline{c}^{\text{ind}}(B) =$

$$\begin{aligned} & \frac{\delta(1-\pi_L)}{2} \cdot \lim_{\pi_H \to 1} (p^{\text{ind}}(GB) - p^{\text{ind}}(BB)) + \frac{\delta}{2}(p^{\text{ind}}(GG) - p^{\text{ind}}(GB) + p^{\text{ind}}(G\varnothing) - p^{\text{ind}}(B\varnothing)) \\ = \; & \frac{\delta(1-\pi_L)^2 \cdot \mu}{4} \left(\frac{\delta}{\mu + (1-\mu)\pi_L^2} + \frac{\delta}{\mu + (1-\mu)\pi_L} \right) \\ := \; & \frac{\delta(1-\pi_L)^2 \cdot \mu}{4} \cdot Y^{\text{ind}}(\mu, \pi_L), \end{aligned}$$

where $Y^{\text{ind}}(\mu, \pi_L) := \frac{\delta}{\mu+(1-\mu)\pi_L^2} + \frac{\delta}{\mu+(1-\mu)\pi_L}$. For a collective brand, $\lim_{\pi_H \to 1} \overline{c}^{\text{col}}(B; I) =$

$$\begin{aligned} & \frac{\delta(1-\pi_L)}{2} \cdot \lim_{\pi_H \to 1} p^{\text{col}}(BG) - p^{\text{col}}(BB) + \\ = \; & \frac{\delta}{2} \cdot ((1+\pi_L) \cdot (p^{\text{col}}(GG) - p^{\text{col}}(GB)) + (1-\pi_L)(p^{\text{col}}(GB) - p^{\text{col}}(BB))) \\ = \; & \frac{\delta(1-\pi_L)^2 \cdot \mu}{4} \cdot Y^{\text{col}}(\mu, \pi_L) \end{aligned}$$

where $Y^{\text{col}}(\mu, \pi_L) = \frac{-2(1+\delta)\mu^3(1-\pi_L)^2 + 2\pi_L(\delta+2\pi_L+3\delta\pi_L) + \mu(2+\delta+4(1+\delta)\pi_L-(10+9\delta)\pi_L^2) - 2\mu^2(1-\pi_L)(4\pi_L+\delta(-1+3\pi_L))}{(2-\mu)(\mu(1-\pi_L)+2\pi_L)(\mu(1+\mu)+2(1-\mu)\mu\pi_L+(2-\mu)(1-\mu)\pi_L^2)}$.
To make a comparison for μ close to 0, it is sufficient to compare Y^{ind} and Y^{col} in that region:

$$\begin{aligned} \lim_{\mu \to 0} Y^{\text{ind}}(\mu, \pi_L) &= \frac{\delta(1+\pi_L)}{\pi_L^2} \\ \lim_{\mu \to 0} Y^{\text{col}}(\mu, \pi_L) &= \frac{2\pi_L + \delta(1+3\pi_L)}{4\pi_L^2} \end{aligned} \tag{25}$$

So, $\lim_{\mu\to 0} Y^{\text{col}} > \lim_{\mu\to 0} Y^{\text{ind}}$ if and only if $\delta < \frac{2\pi_L}{3+\pi_L}$. Thus, by continuity, if $\pi_H = 1$ and μ is close to 0, $\overline{c}^{\text{col}} \geq \overline{c}^{\text{ind}}$ for δ not too large.

For μ close to 1, $\overline{c}^{\text{ind}} \geq \overline{c}^{\text{col}}$ holds for all $\delta \in [0,1]$ because

$$\lim_{\mu\to 1} Y^{\text{ind}}(\mu, \pi_L) = 2\delta > \lim_{\mu\to 1} Y^{\text{col}}(\mu, \pi_L) = \frac{1}{2}\delta(1+\pi_L)$$

for all values of π_L. Therefore, for $\pi_H = 1$ and μ close to 1, an individual brand sustains the reputational equilibrium better. □

Figures and Tables

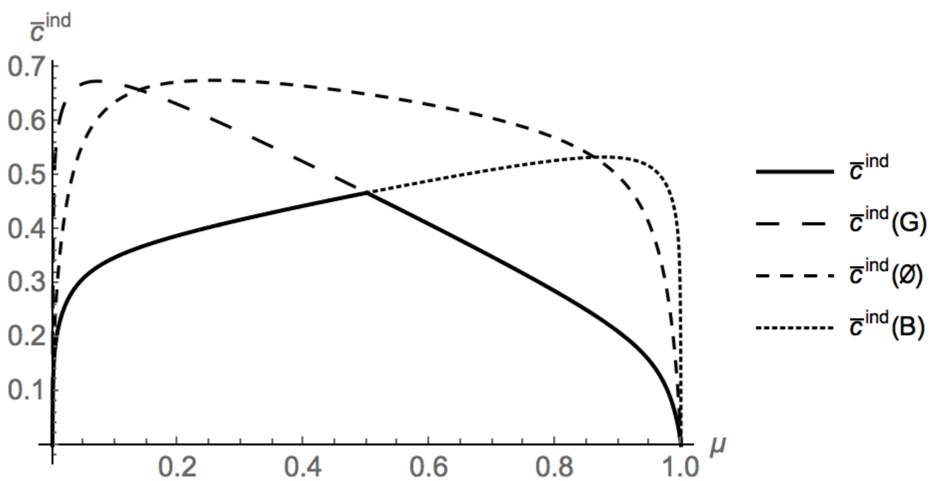


Figure 1: The Threshold Cost for an Individual Brand ($\pi_H = 0.975,\ \pi_L = 0.025,\ \delta = 0.9$)

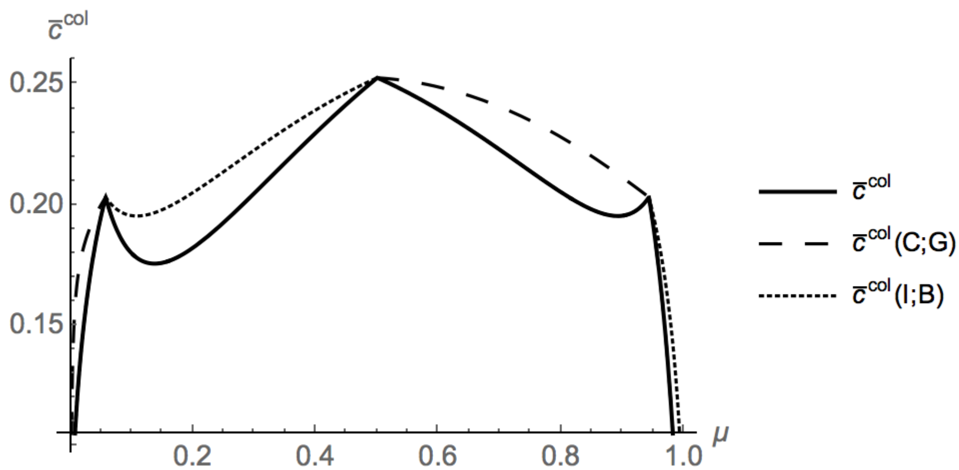


Figure 2: The Threshold Cost for a Collective Brand ($\pi_H = 0.975,\ \pi_L = 0.025,\ \delta = 0.9$)

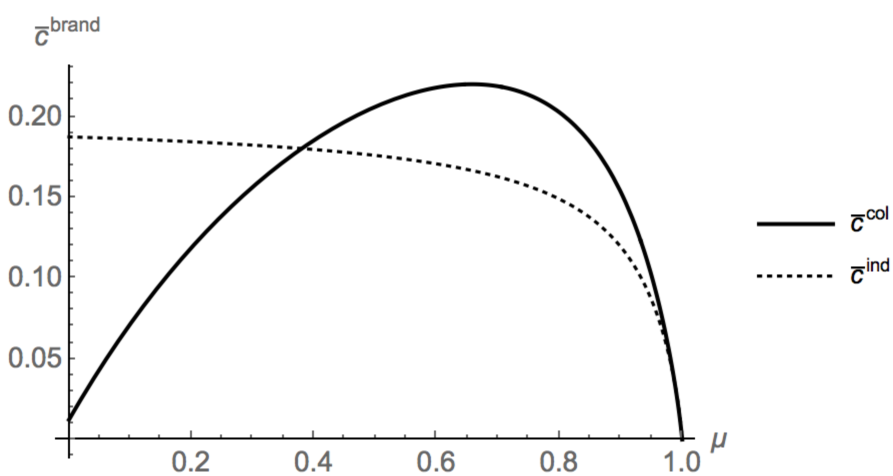
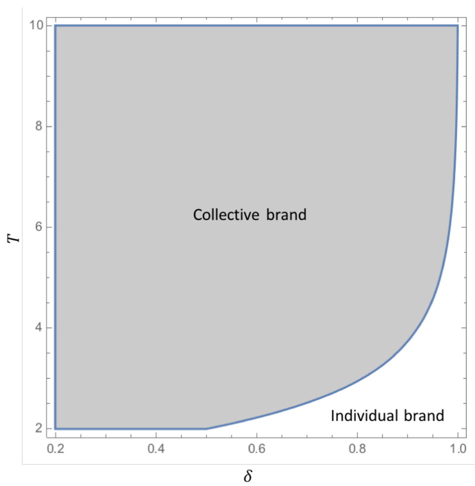
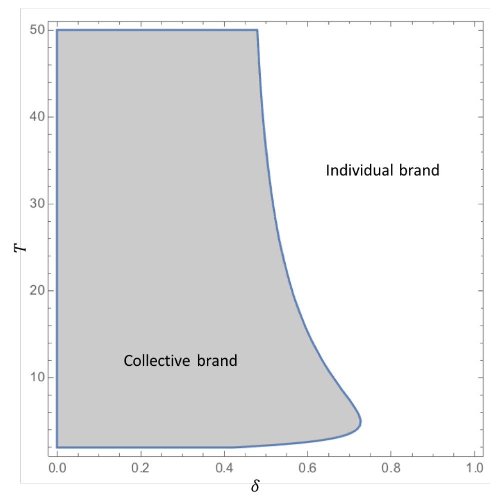


Figure 3: Threshold Costs of Individual ($\overline{c}^{\text{ind}}$) and Collective Brands ($\overline{c}^{\text{col}}$) ($\pi_L = 0, \pi_H = 0.93, \delta = 0.9$)



(a) Exclusive Knowledge Case ($\pi_L = 0$, $\pi_H = 0.8$, $\mu \nearrow 1$)



(b) Quality Control Case ($\pi_H = 1$, $\pi_L = 0.8$, $\mu \searrow 0$)

Figure 4: Comparing the Limit Threshold Costs in the T-δ Space (Shaded if $\overline{c}^{\text{col}} > \overline{c}^{\text{ind}}$)